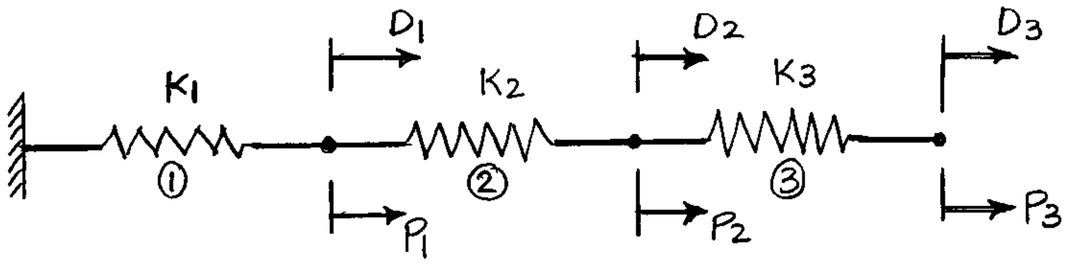
Matrix Analysis of Structures

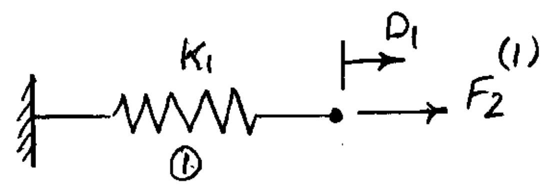
The Stiffness/Displacement Method

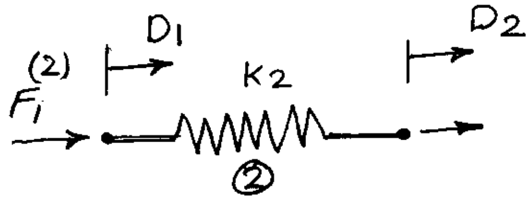
- The Stiffness method is a very powerful analysis technique for formulating the equilibrium equations of a structure interms of Structure Degrees of Freedom and Structure Nodal Forces.
- The Nodal Forces applied at Structure Degrees of Freedom are known quantities, whereas the structure displacements at structure Degrees of Freedom are unknown quantities to be determined.
- The method fulfills the following two primary requirements for an accurate/correct solution:
  - Equilibrium both at the Global Structural Level and at component Level
  - Compatibility of displacements within the structure.
- The method has the advantage of being applicable for both Determinate and Indeterminate Structures
- The method is very appealing because it is very suitable for computer based analysis of structures as a variety of structures can be analyzed following a general algorithm.

Illustrative Example:

We solve a simple structural problem which illustrates the fundamental principles behind the Structure Equilibrium method. We will derive the Equations for the simple structure shown below:







$$K_1 D_1 = F_2^{(1)}$$

Equilibrium Eqn. for Element ①

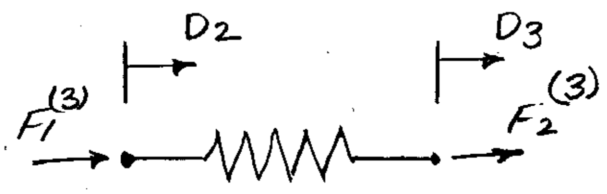
$$K_2 (D_1 - D_2) = F_1^{(2)}$$

$$K_2 (D_2 - D_1) = F_2^{(2)}$$

Equilibrium Eqns. of Element ②

$$K_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{Bmatrix} F_1^{(2)} \\ F_2^{(2)} \end{Bmatrix}$$

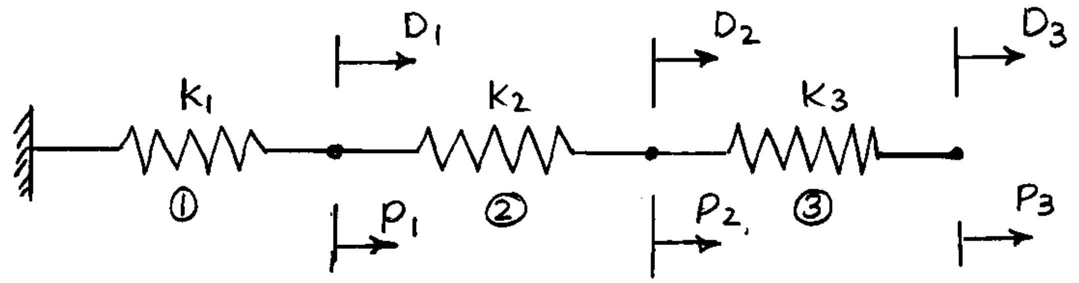
Matrix Form of Equilibrium Eqns for Element ②

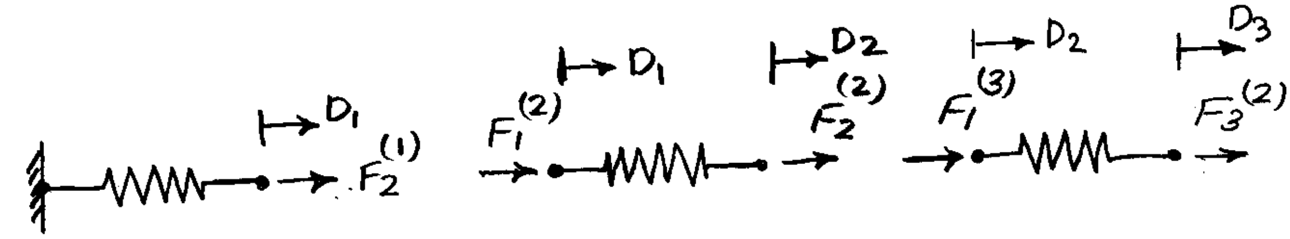


$$K_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} F_1^{(3)} \\ F_2^{(3)} \end{Bmatrix}$$

Equilibrium Eqns. Element ③

Matrix Analysis/Stiffness Method





From Nodal Equilibrium Considerations we can write structure Equilibrium Equations in terms of Element Nodal Forces.

$$\left.\begin{aligned} F_2^{(1)} + F_1^{(2)} &= P_1 \\ F_2^{(2)} + F_1^{(3)} &= P_2 \\ F_2^{(3)} &= P_3 \end{aligned}\right\} \text{— (B)}$$

Substituting in the Nodal Equilibrium Equations (B) the Element Equilibrium Equations we have:

$$K_1 D_1 + K_2 D_1 - K_2 D_2 = P_1$$

$$-K_2 D_1 + K_2 D_2 + K_3 D_2 - K_3 D_3 = P_2$$

$$-K_3 D_2 + K_3 D_3 = P_3$$

$$
\begin{aligned}
(K_1 + K_2) D_1 \; - K_2 D_2 \qquad\qquad\qquad &= P_1 \\
- K_2 D_1 + (K_2 + K_3) D_2 - K_3 D_3 &= P_2 \\
- K_3 D_2 \qquad + K_3 D_3 &= P_3
\end{aligned}
$$

In Matrix Form we have:

$$
\underbrace{\begin{bmatrix} (K_1 + K_2) & -K_2 & 0 \\ -K_2 & (K_2 + K_3) & -K_3 \\ & -K_3 & K_3 \end{bmatrix}}_{\text{Structure Stiffness Matrix}}
\underbrace{\begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix}}_{\substack{\text{Nodal} \\ \text{Displacements}}}
=
\underbrace{\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}}_{\substack{\text{Nodal} \\ \text{Forces}}}
\qquad \text{—— C}
$$

In further Compact Matrix Notation we have:

$$
[K] \{D\} = \{P\} \qquad \text{———— D}
$$

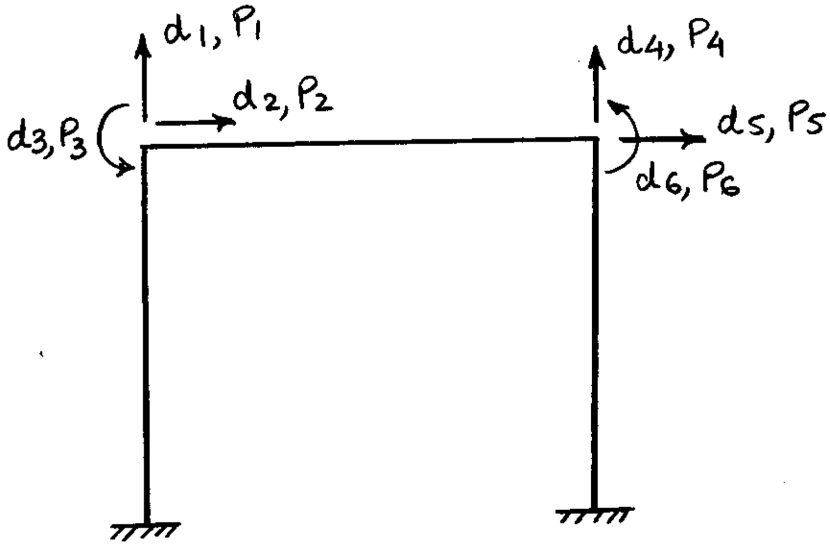
Note! That the structure Stiffness Matrix is symmetric

- Also Note that the Form of Structure Equilibrium Equations shows that the Nodal Forces are expressible as linear combination of Structure nodal displacements.
- Structure Global Equilibrium and Nodal Equilibrium has been imposed through utilizing nodal Equilibrium Equations B
- Nodal compatibility of displacements has been enforced by assuming common displacements at nodal interfaces of elements.

Matrix Analysis - Fundamental Concepts

The response of any structure whether complex or simple can be expressed in the form of simple algebraic equations that relate forces to displacements. If the structure is in a state of equilibrium and small displacement theory is valid then there is a unique relationship between deformations and the loads applied to the structure. The relationship between the loads and displacements can be expressed either through Flexibility coefficients or through Stiffness Coefficients.

Consider the Structure shown below:



If the displacements are $d_1, d_2, d_3 \dots d_6 \dots d_n$ and the forces in the directions of the displacements are $P_1, P_2, P_3, \dots P_6, \dots P_n$

Then the displacements can be expressed as function of forces

$$
\left.
\begin{aligned}
d_1 &= \phi_1(P_1, P_2, \dots P_6) \\
d_2 &= \phi_2(P_1, P_2, \dots P_6) \\
&\;\vdots \\
d_6 &= \phi_6(P_1, P_2, \dots P_6)
\end{aligned}
\right\} \quad \text{—— (A)}
$$

Alternately the Forces can be expressed as a function of displacements.

$$\left.\begin{aligned} P_1 &= \psi_1 (d_1, d_2 \ldots d_6) \\ P_2 &= \psi_2 (d_1, d_2 \ldots d_6) \\ &\vdots \\ P_6 &= \psi_6 (d_1, d_2 \ldots d_6) \end{aligned}\right\} \text{ — (B)}$$

Eqns (A) represent Flexibility relations for the structure and Eqns (B) represent Stiffness relations for the structure.

If the structure is linearly elastic, then there is a linear relationship between the applied nodal loads and the resulting displacements. Thus the Flexibility relations (A) can be written as

$$\left.\begin{aligned} d_1 &= f_{11} P_1 + f_{12} P_2 + \ldots f_{16} P_6 \\ d_2 &= f_{21} P_1 + f_{22} P_2 + \ldots f_{26} P_6 \\ &\vdots \\ d_6 &= f_{61} P_1 + f_{61} P_2 + \ldots f_{66} P_6 \end{aligned}\right\} \text{ — (C)}$$

In Matrix Form

$$\begin{Bmatrix} d_1 \\ d_2 \\ \vdots \\ d_6 \end{Bmatrix} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{14} & f_{15} & f_{16} \\ f_{21} & f_{22} & f_{23} & f_{24} & f_{25} & f_{26} \\ \vdots & & & & & \vdots \\ f_{61} & f_{62} & f_{63} & f_{64} & f_{65} & f_{66} \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_6 \end{Bmatrix} \text{ — } (C_1)$$

or in further Compact Matrix Notation

$$\{d\} = [F]\{P\} \text{ — } (C_2)$$

$\{d\}$ = Vector of Nodal displacements

$[F]$ = Flexibility Matrix

$\{P\}$ = Vector of Nodal Forces.

The Stiffness Relations (B) can be written as:

$$\left.\begin{aligned} P_1 &= K_{11} d_1 + K_{12} d_2 \text{------} K_{16} d_6 \\ P_2 &= K_{21} d_1 + K_{22} d_2 \text{------} K_{26} d_6 \\ &\vdots \\ P_6 &= K_{61} d_1 + K_{62} d_2 \text{------} K_{66} d_6 \end{aligned}\right\} \quad \text{— (D)}$$

In Matrix Form we have

$$\begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_6 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ \vdots & & & & \ddots & \vdots \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ \vdots \\ d_6 \end{Bmatrix} \quad \text{— (D1)}$$

In further compact Matrix Notation

$$\{P\} = [K]\{d\} \quad \text{——— (D2)}$$

$\{P\}$ = Vector of Nodal Forces

$[K]$ = Structure Stiffness Matrix

$\{d\}$ = Vector of Nodal Displacements

Note that the element f_{ij} of the Flexibility Matrix represents displacement at degree of freedom "i" due to unit force at degree of freedom "j"

- Element K_{ij} of the Structure Stiffness Matrix represents force at degree of freedom "i" due to unit force at degree of freedom "j"

- Note that the Flexibility Matrix $[F]$ is symmetric i.e $f_{ij} = f_{ji}$ according to "Maxwell's Theorem of Reciprocal deflections"

It is obvious that the stiffness Matrix $[K]$ is the inverse of the Flexibility Matrix $[F]$ i.e.

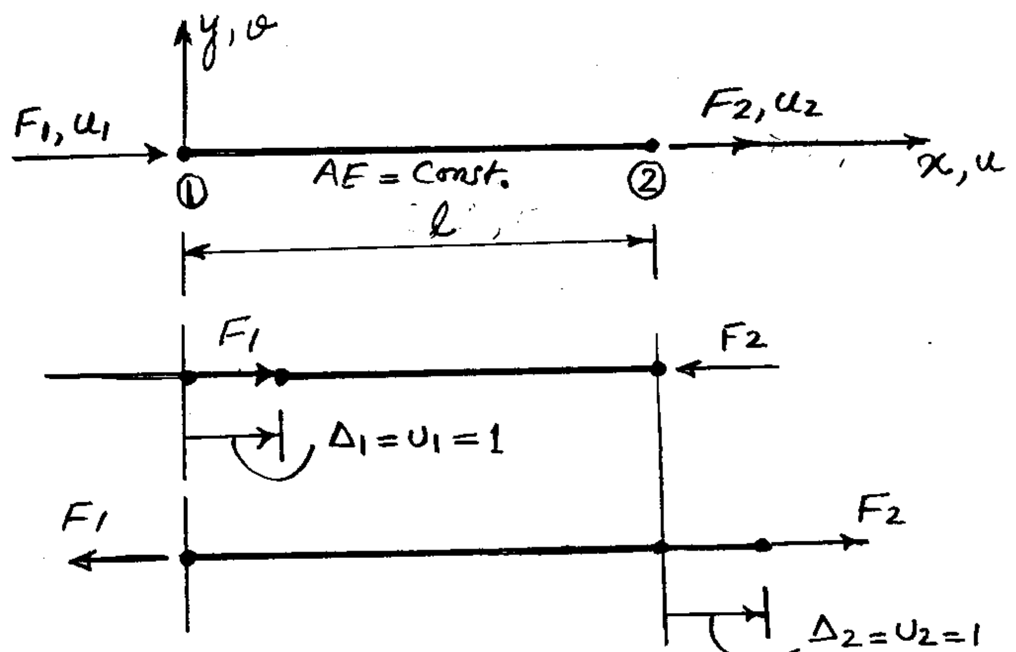
$$[K] = [F]^{-1} \qquad \text{(E)}$$

Since the Flexibility matrix is symmetric, it follows that the Structure Stiffness Matrix is symmetric i.e

$$K_{ij} = K_{ji} \qquad \text{(F)}$$

Stiffness Matrix of a Bar Element

Consider an axial bar element shown below directed along x local axis. It is considered to be a pin-ended member and can therefore resist only axial forces.



We expect the element stiffness matrix to be of the form:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} \qquad \text{(A)}$$

Matrix Analysis - Fondamental Concepts

If we impose a unit displacement at node 1, i.e $\Delta_1 = U_1 = 1$ while restraining displacemend $\Delta_2 = U_2 = 0$, Then

$$\left.\begin{aligned} F_1 &= K_{11} \\ F_2 &= K_{21} \end{aligned}\right\} \text{— (B)}$$

$$\Rightarrow \quad u_1 = \frac{F_1 L}{AE} = 1 \quad \Rightarrow \left.\begin{aligned} F_1 = K_{11} &= \frac{AE}{L} \\ F_2 = K_{21} &= -\frac{AE}{L} \end{aligned}\right\} \text{— (C)}$$

From Statics

Likewise if a unit displacement $U_2 = 1$ is imposed on node 2 along Force F_2, Then

$$\left.\begin{aligned} F_2 &= K_{22} \\ F_1 &= K_{21} \end{aligned}\right\} \text{— (D)}$$

$$u_2 = \frac{F_2 L}{AE} = 1 \quad \Rightarrow \left.\begin{aligned} F_2 = K_{22} &= \frac{AE}{L} \\ F_1 = K_{21} &= -\frac{AE}{L} \end{aligned}\right\} \text{— (E)}$$

From Statics

From (C) and (E) we can write

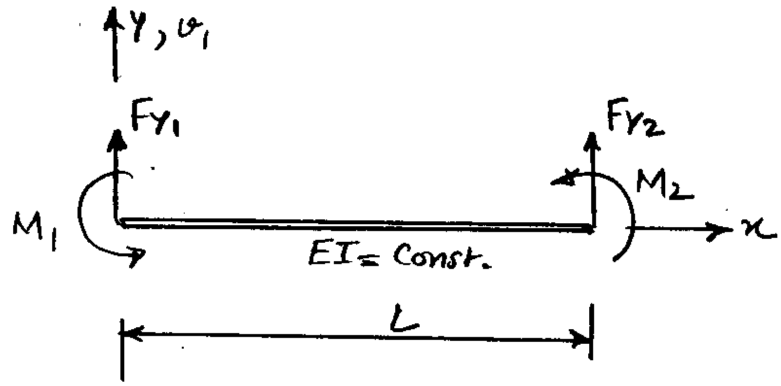
$$\boxed{\begin{aligned} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} &= \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} \\ \text{or} \quad \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} &= \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} \end{aligned}} \text{ — (F)}$$

Stiffness Relations and Stiffness Matrix for a Bar Element.

Note: That the Stiffness matrix is singular (not invertible). This is due to the fact that rigid body modes have not been eliminated. An infinite number of solutions for U_1, U_2 is possible that differ by a rigid body displacement.

Stiffness Matrix for a Beam Element

Consider a Beam Element shown below acted upon by shears and end moments as shown. The stiffness matrix for this element can be developed as follows.

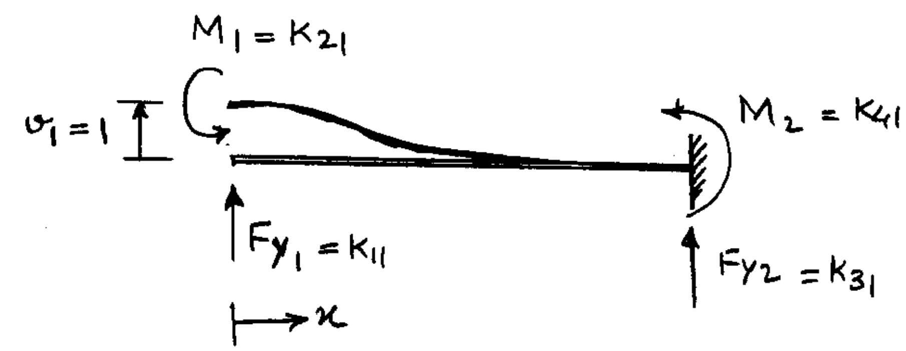


The stiffness Matrix for the element will be of the form:

$$\begin{Bmatrix} F_{Y1} \\ M_1 \\ F_{Y2} \\ M_2 \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

The First Column of the Stiffness matrix can be generated by taking $v_1 = 1$ and $\theta_1 = v_2 = \theta_2 = 0$ and computing the end Forces.

Stiffness Matrix for a Beam Element



$$M = EI\frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = \frac{M}{EI} = \left(-F_{y_1}x - M_1\right) \times \frac{1}{EI}$$

$$\frac{dy}{dx} = \frac{1}{EI}\left(-F_{y_1}\frac{x^2}{2} - M_1 x\right) + C_1$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=0\\x=l}} = 0 \quad , \quad \left.\frac{dy}{dx}\right|_{x=0} = 0 \quad \Rightarrow \quad \boxed{C_1 = 0} \quad \text{—— (A)}$$

$$\left.\frac{dy}{dx}\right|_{x=l} = 0 \quad \Rightarrow \quad -F_{y_1}\frac{l^2}{2} - M_1 l = 0$$

$$\Rightarrow \quad \boxed{F_{y_1} = \frac{2M_1}{l}} \quad \text{—— (B)}$$

Integrate Again

$$y = \frac{1}{EI}\left(-F_{y_1}\frac{x^3}{6} - M_1\frac{x^2}{2}\right) + C_2$$

$$y\Big|_{x=0} = v_1 = 1 \quad , \quad y\Big|_{x=l} = 0$$

$$y\Big|_{x=0} = v_1 = 1 \quad \Rightarrow \quad \boxed{C_2 = 1} \quad \text{—— (C)}$$

Stiffness Matrix For a Beam Element

$$y\Big|_{x=l} = 0$$

$$\Rightarrow \quad \frac{1}{EI}\left(-F_{y1}\frac{l^3}{6} - \frac{M_1 l^2}{2}\right) + 1 = 0$$

$$-F_{y1}\frac{l^3}{6} + M_1\frac{l^2}{2} = EI$$

Substitute

$$F_{y1} = \frac{2M_1}{l} \quad \text{from (B)}$$

$$-2M_1\frac{l^3}{6l} + M_1\frac{l^2}{2} = EI$$

$$M_1 l^2\left(-\frac{1}{3} + \frac{1}{2}\right) = EI$$

$$M_1 l^2\left(\frac{-2+3}{6}\right) = EI$$

$$M_1\frac{l^2}{6} = EI$$

$$\boxed{M_1 = \frac{6EI}{L^2}} \quad \text{—— (D)}$$

From Eqn (B) we have

$$F_{y1} = 2\frac{M_1}{l} = 2\times 6\frac{EI}{L^2}$$

$$\boxed{F_{y1} = 12\frac{EI}{L^2}} \quad \text{—— (E)}$$

It is easy to compute and see that

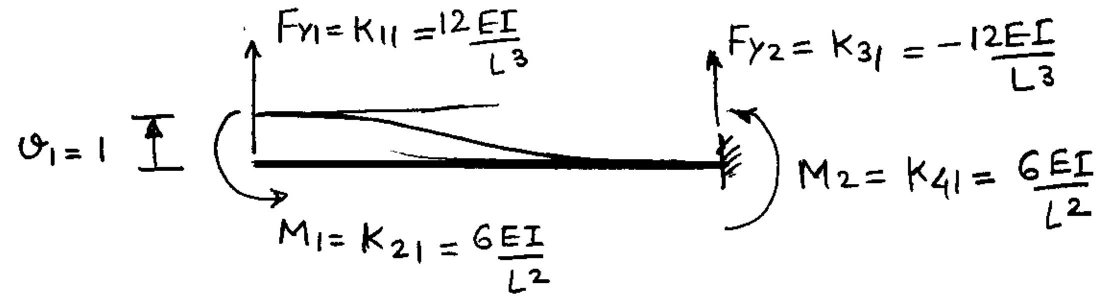
$$\boxed{F_{y2} = -12\frac{EI}{L^2}} \quad \text{—— (F)}$$

$$M_2 = F_{y1}\cdot L - M_1 = 12\frac{EI}{L^2} - \frac{6EI}{L^2}$$

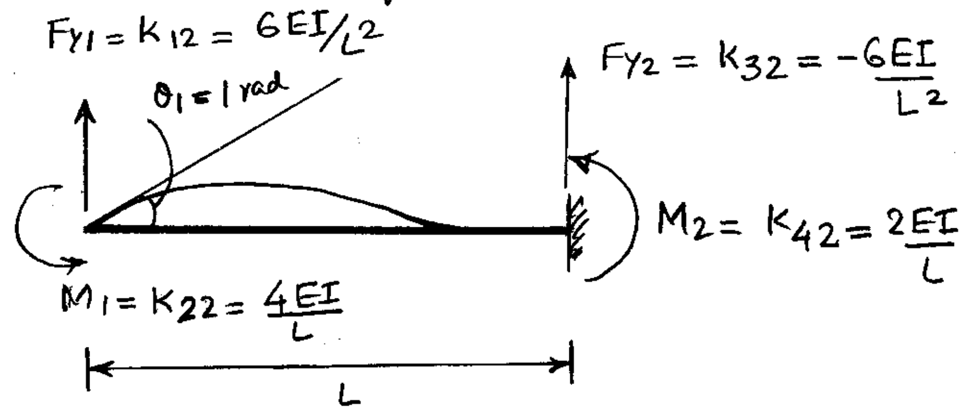
$$\boxed{M_2 = \frac{6EI}{L^2}} \quad \text{—— (G)}$$

Stiffness Matrix for a Beam Element

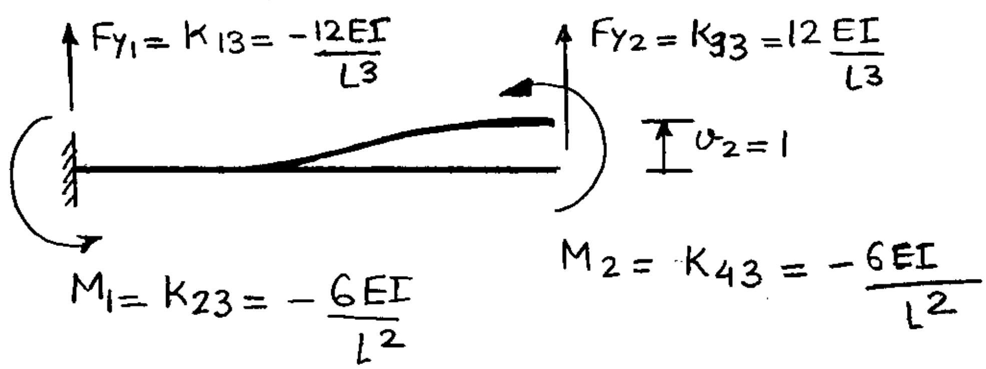
Thus we have generated the First Column/Row of the Element Stiffness Matrix for the Beam Element.



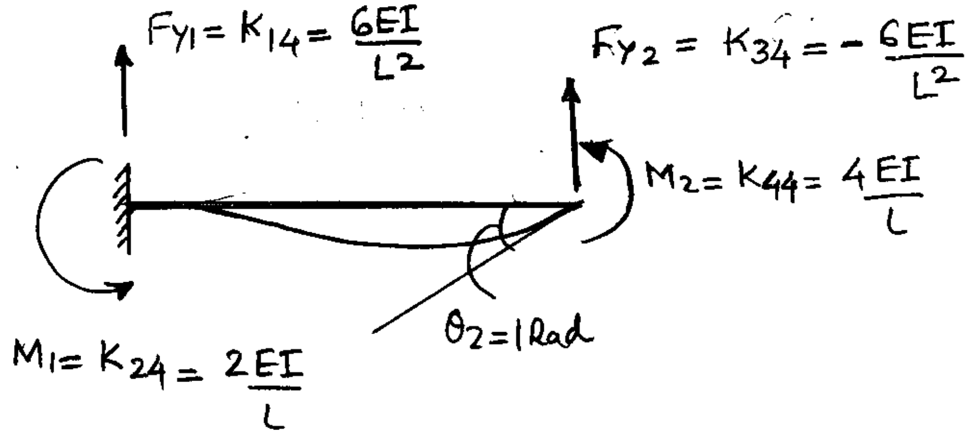
Similarly, to generate the second Column of the Element Stiffness Matrix we impose a unit rotation at End ① and the corresponding generated forces give the second column of the Element Stiffness Matrix.



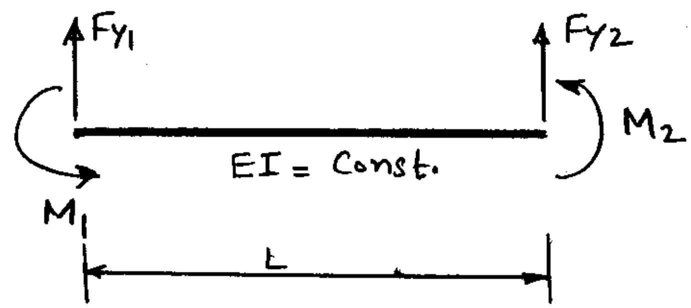
Similarly By Symmetry we can generate the remaining columns of the Element Stiffness Matrix as shown below:



Stiffness Matrix of a Beam Element

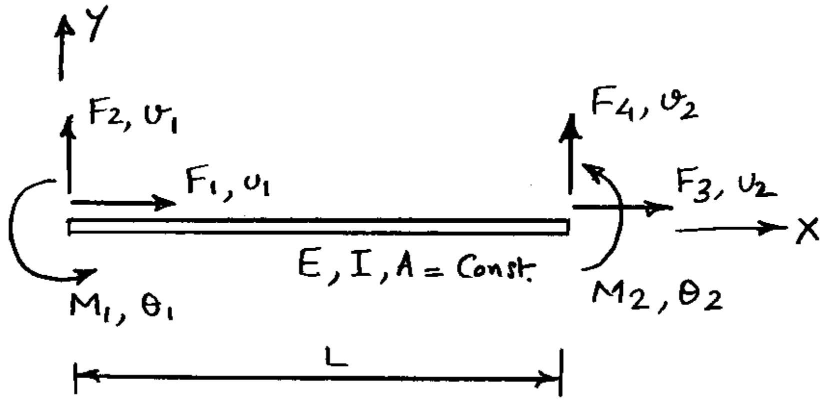


The Element Stiffness Matrix is given below:



$$
\begin{Bmatrix} F_{y1} \\ M_1 \\ F_{y2} \\ M_2 \end{Bmatrix}
=
\underbrace{\begin{bmatrix}
\frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
\frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
\frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}}_{\text{Stiffness Matrix For Prismatic Beam Element.}}
\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}
$$

Stiffness Matrix of a Beam Element



The Stiffness Matrix for the above Beam Element can be easily developed by superposition of the Stiffness Matrices of the Bar Element and the 4 Degree of Freedom Beam Element previously derived. Refer to Bar Element Stiffness Matrix Eqns (E) and Beam Element Stiffness Matrix Eqns (H)

$$\begin{Bmatrix} F_1 \\ F_2 \\ M_1 \\ F_3 \\ F_4 \\ M_6 \end{Bmatrix} = E \begin{bmatrix} \frac{A}{L} & 0 & 0 & -\frac{A}{L} & 0 & 0 \\ 0 & \frac{12I}{L^3} & \frac{6I}{L^2} & 0 & -\frac{12I}{L^3} & \frac{6I}{L^2} \\ 0 & \frac{6I}{L^2} & \frac{4I}{L} & 0 & -\frac{6I}{L^2} & \frac{2I}{L} \\ -\frac{A}{L} & 0 & 0 & \frac{A}{L} & 0 & 0 \\ 0 & -\frac{12I}{L^3} & -\frac{6I}{L^2} & 0 & \frac{12I}{L^3} & -\frac{6I}{L^2} \\ 0 & \frac{6I}{L^2} & \frac{2I}{L} & 0 & -\frac{6I}{L^2} & \frac{4I}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix} \quad \text{— (A)}$$