Methods of Weighted Residuals

Previously we have demonstrated that minimization of Potential Energy Functional Π_p yields the equilibrium equations for a system i.e

$$\delta \Pi_p = 0$$

or

$$\frac{\partial \Pi}{\partial D_i} = 0 \qquad \text{where } D_i = \text{Displacement Variables}$$

yields the governing differential Equations and natural/nonessential Boundary conditions for a structural system.

In many disciplines and areas other than structural mechanics, it is quite common that a variational principle or a functional such as the Potential Energy Functional (Π_p) does not exist or is not known, such as in fluid mechanics.

In such cases the Finite Element Method in conjuctiontion with the Weighted Residuals Method can be used to find a numerical solution provided the governing differential equations of the problem are known.

Methods of Weighted Residuals

Notation

u = Dependent Variables (Displacements etc)

x = Independent Variables (Coordinates of a pt)

f, g = Functions of x, or constants

L, B = Differential Operators

If the Governing Differential Equations and Non-essential boundary conditions of a problem can be expressed in the following form:

$$\left.\begin{aligned} Lu &= f \quad \text{in the Domain } V \\ Bu &= g \quad \text{on the Boundary } S \text{ of } V \end{aligned}\right\} \quad \text{— (1)}$$

The exact solution $u = u(x)$ to the problem is not known. Therefore, we seek an approximate solution $\bar{u}$ to the problem. The approximate solution $\bar{u}$ may be a polynomial that satisfies the essential boundary conditions and contains undetermined coefficients $a_1, a_2, \ldots a_n$ i.e.

$$\bar{u} = \text{Approximate Solution} = \bar{u}(a_i, x). \quad \text{— (2)}$$

The problem then is to find appropriate values of coefficients $(a_1, a_2 \ldots a_n)$ such that $u_{\text{exact}} \simeq \bar{u}_{\text{Approx}}$.

If the approximate solution $\bar{u}$ is substituted in the governing differential equations of the problem then we can expect that there would be some <u>residuals</u> left in the governing differential equations ① as the solution is not exact.

$$\left.\begin{aligned} R_L &= R_L(a, x) = L\bar{u} - f = \text{Interior Residual} \\ R_B &= R_B(a, x) = B\bar{u} - g \quad \text{Boundary Residual} \end{aligned}\right\} ③$$

General Concept/Statement of Weighted Residuals Method

The general concept of Weighted Residual Methods can be mathematically expressed as follows:

$$\left.\begin{aligned} &\int_V W_i R_L \, dv + \int_S W_i R_B \, ds = 0 \\ \text{or} \quad &\int_V W_i (L\bar{u} - f) \, dv + \int_S W_i (B\bar{u} - g) \, ds = 0 \end{aligned}\right\} \text{— } ④$$

In the above statement W_i are the "<u>Weight Functions</u>" $W_i = W_i(x)$ and $\int_V W_i R_L \, dv$, $\int W_i R_B \, ds$ are the "Weighted Averges" of the residuals that have been specified to be zero in an average sense over the domain V and the boundary S of a problem.

Variants of Method of Weighted Residuals

The Galerkin Method

In the Galerkin Method the weight functions used in minimizing the residuals are the shape functions used in the polynomial expression for the approximate solution $\bar{u} = \bar{u}(a_i, x)$ i.e. $W_i = \frac{\partial \bar{u}}{\partial a_i}$

In Structural Mechanics, the residuals are proportional to force or moment, and the shape function can be considered as a virtual displacement or rotation in the Galerkin Statement below:

$$\int_V W_i R_L \, dv + \int_S W_i R_B \, ds = 0$$

$$\int_V W_i (L\bar{u} - f)\, dv + \int_S W_i (B\bar{u} - g)\, ds = 0$$

This virtual work should vanish at an equilibrium configuration. If a varianal functional is available for a problem, both the principle of stationarity of the functional and the Galerkin Method would yield the same result when both these methods employ the same approximating function $\bar{u} = \bar{u}(a_i, x)$

Collocation Method

In collocation method the residual of governing differential equations is set to be zero at certain desired points with in the domain and the boundary of the problem. i.e for "n" different values of x within the domain and on the boundary

$$R_L(a, x_i) = 0 \qquad \text{for } i = 1, 2, 3 \text{----} j-1$$
$$R_B(a, x_i) = 0 \qquad \text{for } i = j, j+1 \text{----} n$$

Least Squares Method

In Least squares Method the coefficients of the approximating function $\bar{u}(a_i, x)$ are so chosen that so as to minimize the integral of the square of the residual. A weight factor $\bar{W}$ is applied to the Boundary Residual R_B. This weight factor can be chosen arbitrarily and may be viewed as a penalty number. A large $\bar{W}$ makes the Boundary Residual R_B more important compared to the Residual R_L over the interior of the domain V.

Least Squares Statement

$$I = \int_V [R_L(a_i, x)]^2 dv + \bar{W}^2 \int_S [R_B(a_i, x)]^2 ds$$

$$\frac{\partial I}{\partial a_i} = 0 \quad , \quad i = 1, 2 \dots n$$

Least Squares Collocation Method

In this method the squares of residuals are calculated at "i" number discrete locations within the domain, where i varies from 1 to m and $m \geq n$. n being the number of terms in the trial function.

Least Squares Collocation Statement:

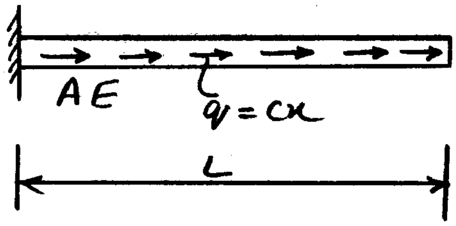
$$I = \sum_{i=1}^{K-1} \left[R_L(a, x_i)\right]^2 + \bar{W}^2 \sum_{i=K}^{m} \left[R_B(a, x_i)\right]^2$$

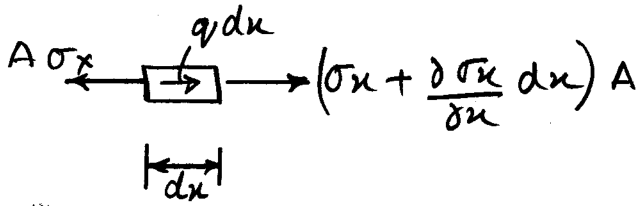
The equations for the coefficients a_i are then obtained by imposing the condition

$$\frac{\partial I}{\partial a_i} = 0 \quad , \quad i = 1, 2 \cdots\cdots n$$

The nature of least squares collocation is such that it yields n equations for a_i even when m (Collocation Pts) $> n$. For $m = n$ the method reduces to simple collocation.

Galerkin Method Example





$$A\sigma_x - \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) A - q\,dx = 0$$

$$\Rightarrow \boxed{A\frac{\partial \sigma_x}{\partial x} + q = 0}$$ — governing differential Equation

For the problem of bar shown above the governing differential equation of equilibrium is also derived and shown above.

The governing differential equation can be cast in terms of displacement as follows

$$\sigma_x = E\,\varepsilon_x = E\frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial \sigma_x}{\partial x} = E\frac{\partial^2 u}{\partial x^2}$$

Substituting above in the governing DE we have

$$AE\frac{\partial^2 u}{\partial x^2} + q = 0$$

or $$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{q}{AE} = 0}$$ — Governing D.E in terms of Displacements. for $0 < x < L$

Example Galerkin Method

If a trial function $\bar{u}$ is selected, the Galerkin statement of the problem can be written as follows:

$$\int_x W_i\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{cx}{AE}\right) dx = 0$$

$$\int_0^L W_i\left(\frac{\partial^2 u}{\partial x^2} + \frac{cx}{AE}\right) dx = 0$$

$$\int_0^L W_i\left(\frac{\partial^2 \bar{u}}{\partial x^2}\right) dx + \int_0^L W_i \frac{cx}{AE}\, dx = 0$$

Integrating the first term of above equation by parts

$$\int u v'\, dx = uv\Big| - \int u' v\, dx$$

$$W_i\left(\frac{\partial u}{\partial x}\right)\Bigg|_0^L - \int_0^L \frac{\partial W_i}{\partial x}\frac{\partial u}{\partial x}\, dx + \int_0^L W_i \frac{cx}{AE}\, dx = 0$$

$$\boxed{\int_0^L \left(-\frac{\partial W_i}{\partial x}\frac{\partial u}{\partial x} + W_i \frac{cx}{AE}\right) dx + W_i \frac{\partial u}{\partial x}\Bigg|_0^L = 0} \quad \text{— (1)}$$

If we select a trial function or the approximating function as

$$\bar{u} = a_1 x + a_2 x^2$$

Then

$$W_1 = \frac{\partial \bar{u}}{\partial a_i} = x, \quad \frac{\partial W_1}{\partial x} = 1$$

$$W_2 = \frac{\partial \bar{u}}{\partial a_2} = x^2, \quad \frac{\partial W_2}{\partial x} = 2x$$

$$\frac{\partial \bar{u}}{\partial x} = a_1 + 2a_2 x$$

Example Galerkin Method

Equation ① can be written as

$$\int_0^L \left(-\frac{\partial W_i}{\partial x} \frac{\partial \bar{u}}{\partial x} + W_i \frac{cx}{AE} \right) dx + W_i \frac{\partial \bar{u}}{\partial x}\Big|_L - W_i \frac{\partial \bar{u}}{\partial x}\Big|_0 = 0 \quad \text{—②}$$

$\boxed{\text{Now} \quad W_i \frac{\partial \bar{u}}{\partial x}\Big|_0 = 0}$ as W_i ' satisfy the essential boundary condition of $\bar{u}\Big|_0 = 0$ at $x = 0$

$W_i \frac{\partial \bar{u}}{\partial x}\Big|_L$ corresponds to the natural boundary condition

at End $x = L$, $\frac{\partial \bar{u}}{\partial x}\Big|_L = \bar{\varepsilon}_x\Big|_L = \frac{\sigma_x}{E}\Big|_L = 0$

With the imposition of Natural Boundary condition $\frac{\partial \bar{u}}{\partial x}\Big|_L = \frac{\sigma_x}{E}\Big|_L = 0$ and the essential Boundary condition $W_i \frac{\partial \bar{u}}{\partial x}\Big|_0 = 0$ The last 2 terms in equation ② Vanish and we have

$$\boxed{\int_0^L \left(-\frac{\partial W_i}{\partial x} \frac{\partial \bar{u}}{\partial x} + W_i \frac{cx}{AE} \right) dx = 0} \quad \text{—— ③}$$

Substituting $\frac{\partial W_i}{\partial x}, \frac{\partial \bar{u}}{\partial x}$ in above eqn we have

$$\int_0^L \left[(-1)(a_1 + 2a_2 x) + x \frac{cx}{AE} \right] dx = 0$$

Example Galerkin Method

$$\int_0^L \left[(-a_1 - 2a_2 x) + \frac{cx^2}{AE} \right] dx = 0$$

$$\left[-a_1 x - a_2 x^2 + \frac{cx^3}{3AE} \right]_0^L = 0$$

$$\Rightarrow \quad -a_1 L - a_2 L^2 + \frac{cL^3}{3AE} = 0$$

$$\boxed{\begin{bmatrix} L & L^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{cL^3}{3AE}} \quad \text{—— (4)}$$

Substitution of W_2, $\frac{\partial W_2}{\partial x}$, $\bar{u}$ in equation 3 yields

$$\int_0^L \left[(-2x)(a_1 + 2a_2 x) + x^2 \frac{cx}{AE} \right] dx = 0$$

$$\left[-a_1 x^2 - 4a_2 x^3 + \frac{cx^4}{4AE} \right]_0^L = 0$$

$$-a_1 L^2 - \frac{4}{3} a_2 L^3 + \frac{cL^4}{4AE} = 0$$

$$\boxed{\begin{bmatrix} L^2 & \frac{4}{3} L^3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{cL^4}{4AE}} \quad \text{—— (5)}$$

Eqns (4) & (5) can be written in matrix form as

$$\boxed{\begin{bmatrix} 1 & L \\ L & \frac{4}{3} L^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{c}{AE} \begin{Bmatrix} L^2/3 \\ L^3/4 \end{Bmatrix}} \quad \text{— (6)}$$

Example Galerkin Method

Solution of simultaneous system of equations (6) gives for coefficients a_1 and a_2

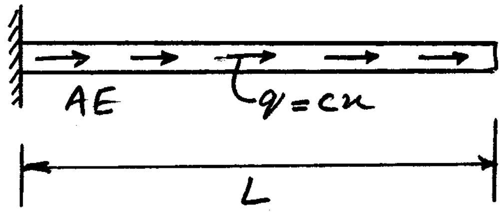
$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{7cL^2}{12AE} \\ -\dfrac{cL}{4AE} \end{Bmatrix}$$

And the trial function $\bar{u} = a_1 x + a_2 x$ is now given as

$$\boxed{\bar{u} = \frac{7cL^2}{12AE}x - \frac{cL}{4AE}x^2} \quad \text{—— (7)}$$

This expression for $\bar{u}(a_i, x)$ is the same as that derived previously for this bar problem using the Rayleigh-Ritz method

Example Collocation Method



We now solve the same bar problem using point collocation method

The Differential Equation for the problem is

$$\frac{\partial^2 u}{\partial x^2} + \frac{cx}{AE} = 0 \quad \text{for } 0 < x < L$$

Boundary Condition

$$\left.\frac{\partial u}{\partial x}\right|_{x=L} = 0$$

Trial Function is

$$\bar{u} = a_1 x + a_2 x^2$$

which satisfies the essential Boundary Condition $u\big|_{x=0} = 0$

Interior Residual is

$$R_L = L\bar{u} - f = 2a_2 + \frac{cx}{AE}$$

Boundary Residual is

$$R_B = B\bar{u} - g = \frac{\partial u}{\partial x} = a_1 + 2a_2 x\Big|_L = a_1 + 2a_2 L$$

For Collocation Solution we arbitrarily evaluate the interior Residual R_L at $\frac{L}{3}$ and set it to zero

$$R_L\Big|_{L/3} = 2a_2 + \frac{cL}{3AE} = 0 \quad \Rightarrow \boxed{a_2 = -\frac{cL}{6AE}}$$

The Boundary Residual R_B of course is evaluated at the Boundary at $x = L$ and is set to zero

$$R_B\Big|_L = a_1 + 2a_2 L = 0$$

$$= a_1 + 2\left(-\frac{cL}{6AE}\right)L = 0 \quad \Rightarrow \boxed{a_1 = \frac{cL^2}{3AE}}$$

Example Collocation Method

The Collocation method based solution is then

$$\boxed{\bar{u} = \frac{cL^2}{3AE}x - \frac{cL}{6AE}x^2}$$

Example Least Squares Method

Least squares statement

$$I = \int_V \left[R_L(a_i, x)\right]^2 dv + \bar{W}^2 \int_S \left[R_B(a_i, x)\right]^2 ds$$

and

$$\frac{\partial I}{\partial a_i} = 0$$

For $\bar{u} = a_1 x + a_2 x^2$

$$R_L = L\bar{u} - f = \frac{\partial^2 \bar{u}}{\partial x^2} + \frac{cx}{AE} = 2a_2 + \frac{cx}{AE}$$

$$R_B = B\bar{u} - g = \frac{\partial \bar{u}}{\partial x} = a_1 + 2a_2 x \Big|_{x=L} = a_1 + 2a_2 L$$

Then

$$I = \int_0^L \left(2a_2 + \frac{cx}{AE}\right)^2 dx + \bar{W}^2 \left[a_1 + 2a_2 L\right]^2$$

In above expression the term a_2 will have same dimensions if we use $\bar{w}^2 = \frac{1}{L}$

Hence

$$I = \int_0^L \left(2a_2 + \frac{cx}{AE}\right)^2 dx + \frac{1}{L}\left[a_1 + 2a_2 L\right]^2$$

Example Least Squares Method

$$I = \int_0^L \left(4a_2^2 + \frac{c^2x^2}{(AE)^2} + 4a_2\frac{cx}{AE}\right)dx + \frac{1}{L}(a_1 + 2a_2L)^2$$

$$I = 4a_2^2x + \frac{c^2x^3}{3(AE)^2} + 2a_2\frac{cx^2}{AE}\Bigg|_0^L + \frac{1}{L}(a_1 + 2a_2L)^2$$

$$I = 4a_2^2L + \frac{c^2L^3}{3(AE)^2} + 2a_2\frac{cL^2}{AE} + \frac{1}{L}(a_1 + 2a_2L)^2$$

$$\frac{\partial I}{\partial a_1} = \frac{2}{L}(a_1 + 2a_2L) = 0$$

$$\Rightarrow \boxed{\frac{\partial I}{\partial a_1} = a_1 + 2a_2L = 0 \Rightarrow a_1 = -2a_2L} \quad \text{— ①}$$

$$\frac{\partial I}{\partial a_2} = 8a_2L + 2\frac{cL^2}{AE} + \frac{2}{L}(a_1 + 2a_2L)\,2L = 0$$

$$= 8a_2L + 2\frac{cL^2}{AE} + 4a_1 + 8a_2L = 0$$

$$\boxed{\frac{\partial I}{\partial a_2} = 4a_1 + 16a_2L + 2\frac{cL^2}{AE} = 0} \quad \text{— ②}$$

From ① and ② we have

$$4(-2a_2L) + 16a_2L + 2\frac{cL^2}{AE} = 0$$

$$-8a_2L + 16a_2L + 2\frac{cL^2}{AE} = 0$$

$$a_2 = -\frac{2cL^2}{AE} \times \frac{1}{8L} = -\frac{cL}{4AE}$$

$$\boxed{\begin{aligned} a_2 &= -\frac{cL}{4AE} \\ a_1 &= -2a_2L = -2\left(\frac{-cL}{4AE}\right)L = \frac{cL^2}{2AE} \end{aligned}}$$

Example Least Squares Method

The expression for $\bar{u}$ then becomes

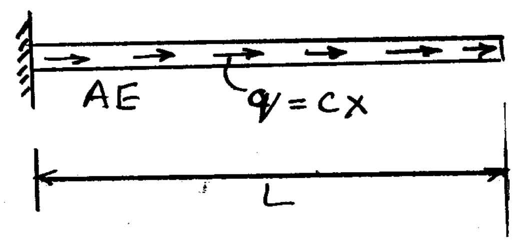
$$\boxed{\begin{aligned} \bar{u} &= \frac{cL^2 x}{2AE} - \frac{cL}{4AE}x^2 \\ \bar{\sigma}_x &= \frac{cL^2}{2AE} - \frac{2cL}{2AE}x \end{aligned}}$$

Solution

Example Least Squares Collocation Method

Differential Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{cx}{AE} = 0 \,, \quad 0 < x < L$$



Boundary Condition

$$E \left. \frac{\partial u}{\partial x} \right|_{x=L} = 0$$

Trial Function

$$\bar{u} = a_1 x + a_2 x^2$$

For Least Squares Collocation we elect to evaluate Residuals at $x = \frac{L}{3}$ and at $x = L$. The boundary residual is also evaluated at $x = L$

Interior Residual is

$$R_L = L\bar{u} - f = 2a_2 + \frac{cx}{AE}$$

$$R_B = B\bar{u} - g = \frac{\partial u}{\partial x} = \left. a_1 + 2a_2 x \right|_L = a_1 + 2a_2 L$$

$$R_1 = \left. R_L \right|_{x = L/3} = 2a_2 + \frac{cL}{3AE}$$

$$R_2 = \left. R_L \right|_{x = L} = 2a_2 + \frac{cL}{AE}$$

$$R_B = \left. R_B \right|_{x=L} = a_1 + 2a_2 L$$

Example Least Squares Collocation

For dimensional homogenicty weighting function for the boundary residual is chosen as $\bar{W}^2 = \frac{1}{L^2}$

Since we still have to take square of the residuals, this is accomplished by dividing the Boundary Residual R_B by L

Thus

$$R_B/L = \frac{a_1}{L} + 2a_2$$

These residuals can be written in Matrix Form as follows:

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_B/L \end{Bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \\ 1/L & 2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} - \begin{Bmatrix} -cL/3AE \\ -cL/AE \\ 0 \end{Bmatrix}$$

This in matrix notation is expressed as:

$$\{R\} = [Q]\{a\} - \{b\}$$

For least squares solution we need to calculate

$$I = R_1^2 + R_2^2 + R_B^2/L^2$$

$$I = \{R\}^T\{R\} = (Qa - b)^T(Qa - b)$$

$$= (a^TQ^T - b^T)(Qa - b)$$

$$= a^TQ^TQa - a^TQ^Tb - b^TQa + b^Tb$$

$$I = a^TQ^TQa - 2a^TQ^Tb + b^Tb$$

Example Least Squares Collocation

Applying the Condition

$$\left\{\frac{\partial I}{\partial a}\right\} = \{0\} \quad \text{yields}$$

$$\boxed{\left\{\frac{\partial I}{\partial a}\right\} = Q^T Q\, a - Q^T b = 0} \qquad \text{— (1)}$$

The above equation should yield the required values of coefficients a_i

$$Q^T Q = \begin{bmatrix} 0 & 0 & 1/L \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 2 \\ 1/L & 2 \end{bmatrix} = \begin{bmatrix} 1/L^2 & \frac{2}{L} \\ \frac{2}{L} & 12 \end{bmatrix}$$

Symmetric

$$Q^T b = \begin{bmatrix} 0 & 0 & \frac{1}{L} \\ 2 & 2 & 2 \end{bmatrix} \begin{Bmatrix} -cL/3AE \\ -cL/AE \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\frac{8cL}{3AE} \end{Bmatrix}$$

Substituting the above computed Matrices in equation (1) above we have

$$\underbrace{\begin{bmatrix} 1/L^2 & \frac{2}{L} \\ 2/L & 12 \end{bmatrix}}_{A} \underbrace{\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}}_{x} = \underbrace{\begin{Bmatrix} 0 \\ -\frac{8cL}{3AE} \end{Bmatrix}}_{B}$$

$$A^{-1} = \frac{1}{\text{Det}A} \text{Adj}A =$$

$$\text{Det } A = \frac{12}{L^2} - \frac{4}{L^2} = \frac{8}{L^2}$$

Example Least Squares Collocation

$$\text{Adj } A = \begin{bmatrix} 12 & -\frac{2}{L} \\ -\frac{2}{L} & \frac{1}{L^2} \end{bmatrix}$$

$$\Rightarrow \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{L^2}{8} \begin{bmatrix} 12 & -\frac{2}{L} \\ -\frac{2}{L} & \frac{1}{L^2} \end{bmatrix} \begin{Bmatrix} 0 \\ -\frac{8cL}{3AE} \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \frac{2cL^2}{3AE} \\ -\frac{cL}{3AE} \end{Bmatrix}$$

And we finally have our expression for $\bar{u}$ as

$$\bar{u} = a_1 x + a_2 x^2 =$$

$$\boxed{\begin{aligned} \bar{u}(x) &= \frac{2cL^2}{3AE}x - \frac{cLx^2}{3AE} \\ \bar{\sigma}_x &= \frac{\partial \bar{u}}{\partial x}\cdot E = \frac{2cL^2}{3A} - \frac{2cLx}{3A} \end{aligned}}$$

Solution.

Comparison of Various Methods of Weighted Residuals

Exact Solution

$$u = \frac{cL^2}{2AE}x - \frac{c}{6AE}x^3 \qquad \equiv \qquad u = \frac{cL^3}{2AE}\left(\xi - \frac{\xi^3}{3}\right)$$

$$\sigma x = E\frac{\partial u}{\partial x} = \frac{cL^2}{2A} - \frac{c}{2A}x^2 \qquad \sigma x = \frac{cL^2}{2A}\left(1 - \xi^2\right)$$

Galerkin Method

$$\bar{u} = \frac{7cL^2}{12AE}x - \frac{cL}{4AE}x^2 \qquad \equiv \qquad \bar{u} = \frac{cL^3}{2AE}\left(\frac{7}{6}\xi - \frac{\xi^2}{2}\right)$$

$$\bar{\sigma} x = \frac{7cL^2}{12A} - \frac{cL}{2A}x \qquad \bar{\sigma} x = \frac{cL^2}{2A}\left(\frac{7}{6} - \xi\right)$$

Collocation Method

$$\bar{u} = \frac{cL^2}{3AE}x - \frac{cL}{6AE}x^2 \qquad \equiv \qquad \bar{u} = \frac{cL^3}{2AE}\left(\frac{2}{3}\xi - \frac{1}{3}\xi^2\right)$$

$$\bar{\sigma} x = \frac{cL^2}{3A} - \frac{cL}{3A}x \qquad \bar{\sigma} x = \frac{cL^2}{2A}\left(\frac{2}{3} - \frac{2}{3}\xi\right)$$

Least Squares Method

$$\bar{u} = \frac{cL^2}{2AE}x - \frac{cL}{4AE}x^2 \qquad \equiv \qquad \bar{u} = \frac{cL^3}{2AE}\left(\xi - \frac{1}{2}\xi^2\right)$$

$$\bar{\sigma} x = \frac{cL^2}{2AE} - \frac{cL}{2AE}x \qquad \bar{\sigma} x = \frac{cL^2}{2A}\left(1 - \xi\right)$$

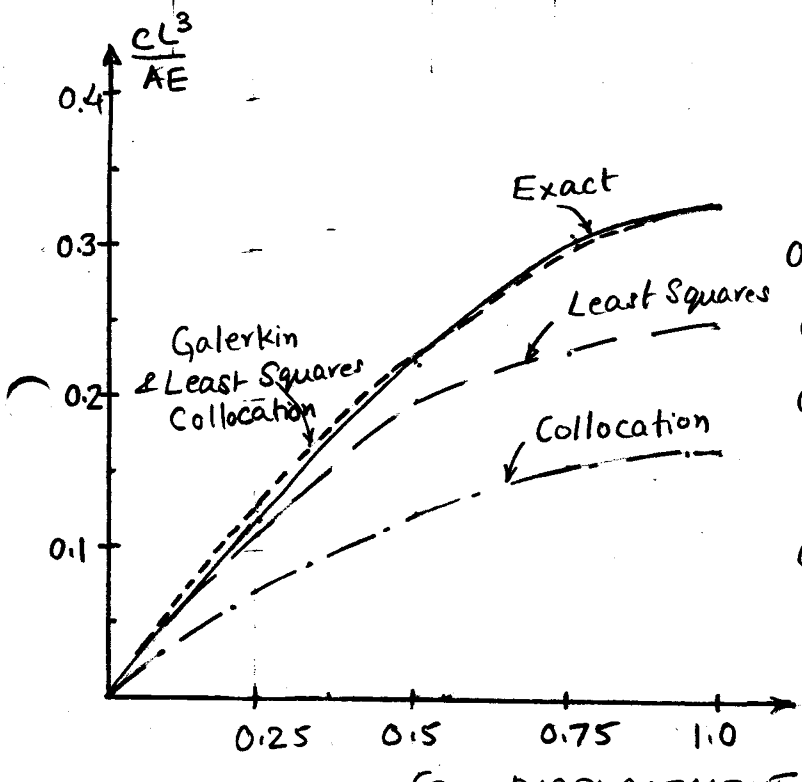
Least Squares Collocation

$$\bar{u} = \frac{2cL^2}{3AE}x - \frac{cL}{3AE}x^2 \qquad \equiv \qquad \bar{u} = \frac{cL^3}{2AE}\left(\frac{4}{3}\xi - \frac{2}{3}\xi^2\right)$$

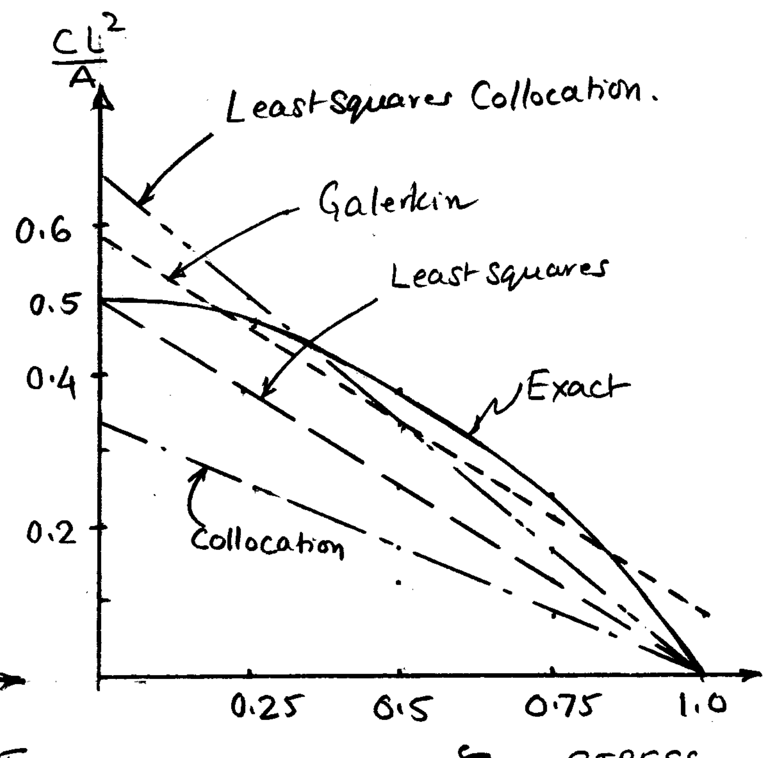
$$\bar{\sigma} x = \frac{2cL^2}{3A} - \frac{2cL}{3A}x \qquad \bar{\sigma} x = \frac{cL^2}{2A}\left(\frac{4}{3} - \frac{4}{3}\xi\right)$$

Comparison of Various Methods of Weighted Residuals

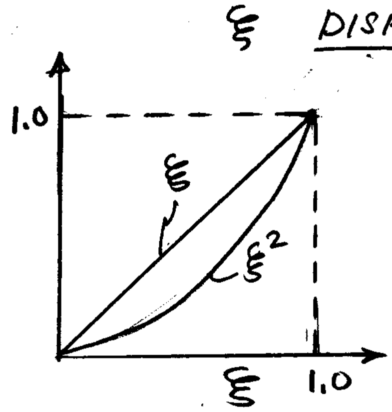
Coord ξ	Exact		Galerkin		Collocation		Least Squares		Least Squares Collocation	
	u	σ_x	u	σ_x	u	σ_x	u	σ_x	u	σ_x
0	0	0.5	0	0.5833	0	0.3333	0	0.50	0	0.667
0.25	0.1224	0.4688	0.1302	0.4583	0.0729	0.250	0.1094	0.375	0.1458	0.500
0.50	0.2292	0.3750	0.2292	0.3333	0.1250	0.1667	0.1875	0.250	0.250	0.333
0.75	0.3047	0.2187	0.2968	0.2083	0.1563	0.0833	0.2343	0.125	0.3125	0.1667
1.0	0.3333	0	0.3333	0.0833	0.1667	0	0.25	0	0.3333	0



DISPLACEMENT



STRESS



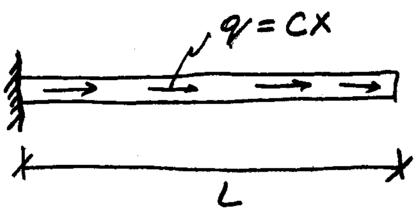
SHAPE FUNCTIONS

Some Comments About Variational Forms, Methods of Weighted Residuals and Boundary Conditions

For a problem if the highest order of derivaties appearing in the governing differential equations of the problem is "2m", then the highest order of the derivatives appearing in the Variational Form will be "m". Essential Boundary Conditions will involve derivatives of the order "zero" to "m-1", the zeroth derivative being the dependend variable itself. The non-essential Boundary Conditions (Force Boundary Conditions) involve derivatives of order "m" and higher upto "2m-1."

Example

Bar Problem



Differential Equation:
$$AE \frac{\partial^2 u}{\partial x^2} + cx = 0$$

Then

highest order derivative in D.E $= 2m = 2$

$\Rightarrow$ Highest order derivative in the Variational Fonctional will be: $m = 1$

$$\Pi_P = U - W = \int_0^L \frac{EA}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx - \int_0^L cx\, u\, dx$$

Highest Order of Derivative on Essential BC:

0 to $m-1 = 0$ ie 0 on "u" itself

ie $u\big|_x = 0$ at $x = 0$

Highest Order derivative on Natural/Force Boundary Conditions:

m to $2m+1$

ie 1 to $2-1=1 \Rightarrow 1$

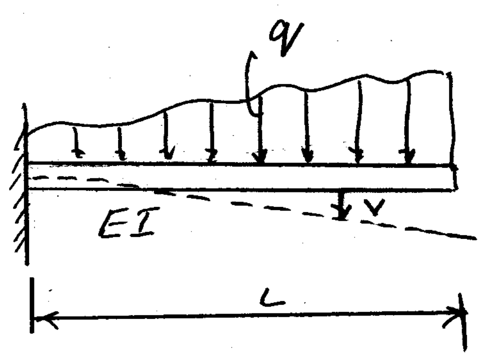
Force B.C

$\sigma x = 0$ at $x = L$

or $E \dfrac{\partial u}{\partial x}\Big|_L = 0$

Example (2)

Beam Bending Problem



Differential Equation

$$EI \frac{\partial^4 v}{\partial x^4} - q = 0$$

Highest order derivative in Diff. Eqn $= 2m = 4$

Highest order derivative in Variational Functional: $= m = 2$

$$\Pi p = \frac{1}{2}\int_0^L EI\left(\frac{\partial^2 v}{\partial x^2}\right)^2$$

Highest Order Derivative in Essential Boundary Condition $= 0 \rightarrow m-1$

ie $0 \rightarrow 1$

$$v\Big|_{x=0} = 0 \quad , \quad \frac{\partial v}{\partial x}\Big|_{x=0} = 0$$

Highest order Derivative in Force Boundary Conditions

$m \rightarrow 2m-1 \Rightarrow 2 \rightarrow 3$

$$M\Big|_{x=L} = EI\frac{\partial^2 v}{\partial x^2}\Big|_{x=L} = 0 , \quad V\Big|_{x=L} = EI\frac{\partial^3 v}{\partial x^3}\Big|_{x=L} = 0$$

Some Comments about Methods of Weighted Residuals

General Statement of Method of Weighted Residuals

$$\int_V W_i R_L \, dv + \int_S W_i R_B \, ds = 0$$

- In Galerkin Method the weight functions W_i are the shape functions being used in the trial function $\bar{u}(a_i, x)$
- In the Collocation method the weight function W_i is a unit delta function that is non-zero only at the collocation points.
- In the least squares method the weight functions are $W_i = \dfrac{\partial R}{\partial a_i}$
- All methods of weighted residuals yield equations of the form

$$[A]\{a\} = \{b\}$$ from {a} coefficients of shape functions in the trial function can be determined.

- Galerkin Method and least squares methods yield symmetric matrix [A], wheras collocation method may produce non-symmetric coefficient matrix [A]
- Usage of integration by parts in Galerkin method reduces the continuity requirements on the trial function $\bar{u}(a_i, x)$, whereas the least squares method does not benefit from integration by parts.
- Least squares method may lead to ill-conditioned Coefficient Matrix [A].