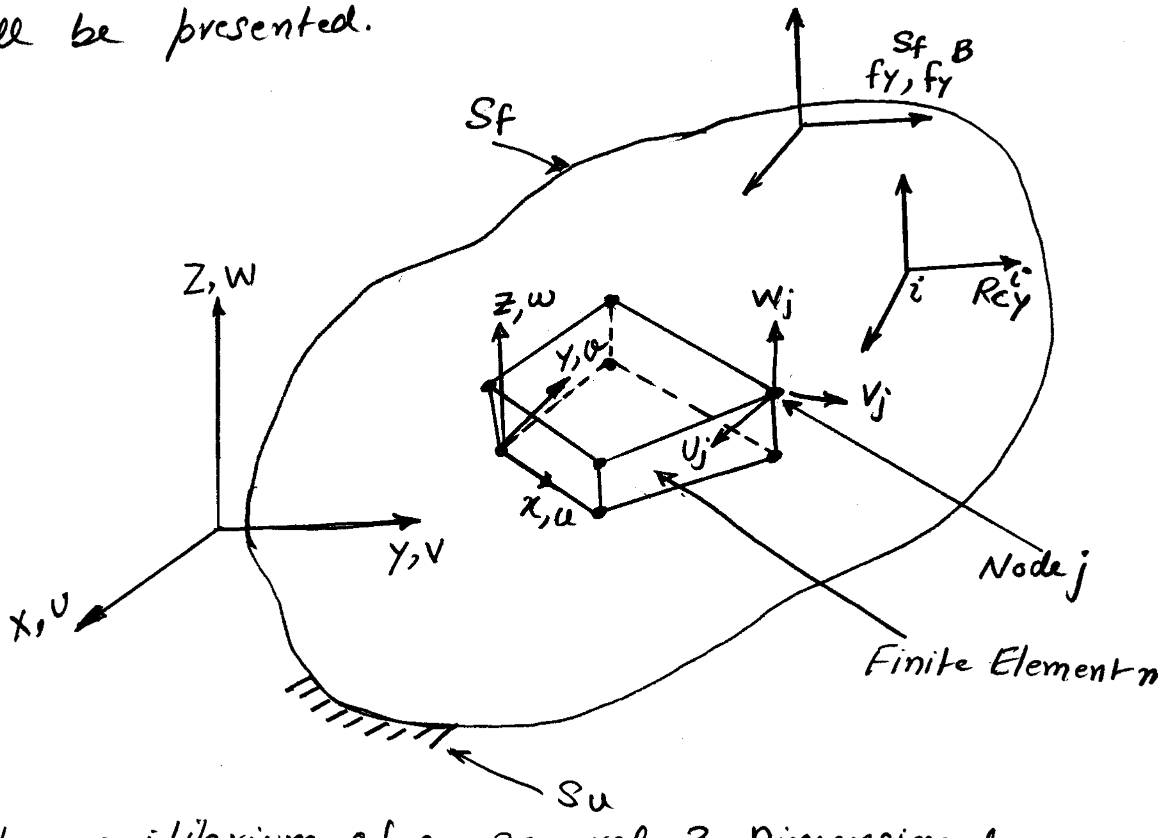
General Derivations of Finite Element Equilibrium Equations

In this section we first present the principle of virtual displacement as it would apply to a 3-Dimensional Continuum in a theory of elasticity problem.

Later the general application of the principle of virtual work in the context of finite element method will be presented and application of the principle of virtual displacements to yield equilibrium equations of a general system that has been discretized into finite elements shall be presented.



Consider the equilibrium of a general 3-Dimensional body as shown in the figure. The body is located in a fixed (stationary) Coordinate system X, Y, Z. The body is supported on area Su with prescribed displacements U^{Su} and is subjected to surface tractions f^{Sf} (forces per unit area) on the surface area.

In addition the body is subjected to externally applied body forces f^B (forces per unit volume) and concentrated loads R_C^i (i denotes point of application of load)

The body forces f^B, Surface tractions f^{Sf} and concentrated forces R_C^i in general would have three components corresponding to Global X, Y, Z coordinate axes:

$$f^B = \begin{Bmatrix} f_X^B \\ f_Y^B \\ f_Z^B \end{Bmatrix} \quad , \quad f^{Sf} = \begin{Bmatrix} f_X^{Sf} \\ f_Y^{Sf} \\ f_Z^{Sf} \end{Bmatrix} \quad , \quad R_C^i = \begin{Bmatrix} R_{CX}^i \\ R_{CY}^i \\ R_{CZ}^i \end{Bmatrix}$$

The displacements of the body from the unloaded configurations are also measured in the Global X, Y, Z coordinate system and are denoted by U

$$U(X, Y, Z) = \begin{Bmatrix} U \\ V \\ W \end{Bmatrix}$$

and the displacements on the supporting surface S_u are denoted by $U = U^{Sf}$

The strains corresponding to Displacements U are

$$\varepsilon^T = [\varepsilon_{XX} \; \varepsilon_{YY} \; \varepsilon_{ZZ} \; \gamma_{XY} \; \gamma_{YZ} \; \gamma_{XZ}]$$

where $$\varepsilon_{XX} = \frac{\partial U}{\partial X}, \quad \varepsilon_{YY} = \frac{\partial V}{\partial Y}, \quad \varepsilon_{ZZ} = \frac{\partial W}{\partial Z}$$

$$\gamma_{XY} = \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}, \quad \gamma_{YZ} = \frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}, \quad \gamma_{XZ} = \frac{\partial W}{\partial X} + \frac{\partial U}{\partial Z}$$

The stresses corresponding to ε are

$$\sigma^T = [\sigma_{xx}\ \sigma_{yy}\ \sigma_{zz}\ \sigma_{xy}\ \sigma_{yz}\ \sigma_{xz}]$$

where

$$\sigma = C\varepsilon + \sigma^i$$

C is the material stiffness matrix relating strains ε to stresses σ and σ^i denotes the initial stresses.

In the problem posed we assume that:

- The displacements are small so that the strains calculated from the displacements by relations

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad ---- \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

are valid.

- The material stiffness matrix C can vary as a function of x, y, z but is constant otherwise ie C does not depend upon stress state.

General Derivation of Finite Element Equilibrium Equations

Application of Principle of Virtual Displacements

The displacement based Finite Element Method is based on the Principle of Virtual Displacements. or the Principle of Virtual Work. According to the Principle of Virtual Work "Equilibrium of a body requires that for any compatible small virtual displacements which satisfy the essential/imposed displacement boundary conditions and which are imposed on the body in its state of equilibrium, the total internal virtual work done is equal to the total external virtual work done." *

Internal Virtual Work

External Virtual Work

$$\int_V \bar{\varepsilon}^T \tau \, dv = \int_V \bar{U}^T f^B dv + \int_{S_f} \bar{U}^{S_f T} f^{S_f} ds + \sum_i \bar{U}^{i T} R_c^i \qquad \text{(A)}$$

f^B, f^{S_f}, R_c^i: Applied Loads

τ: Stresses in Equilibrium with Applied Loads

$\bar{\varepsilon}$: Virtual Strains corresponding to virtual displacements $\bar{U}$

General Derivation of Finite Element Equations from Principle of Virtual Work

If the 3-Dimensional Body considered previously is discretized into an assemblage of finite elements, then the displacements measured in the local coordinate system x, y, z within each element are assumed to be a function of the displacements at the "N" finite element nodal points. For a typical element "m" we can write:

$$u^{(m)}(x,y,z) = H^{(m)}(x,y,z)\,\hat{U} \qquad \text{———} \quad (1)$$

where $H^{(m)}$ = Displacement Interpolation Matrix for element m

$\hat{U}$ = Vector of three global displacement components U_i, V_i, and W_i at all the nodal points including the supports of dimension 3 N

$$\hat{U}^T = [U_1\ V_1\ W_1\ .\ U_2\ V_2\ W_2\ ----\ U_N\ V_n\ W_n]$$

$$\text{or } \hat{U}^T = [U_1\ U_2\ U_3\ -----\ U_N]$$

From Equation (1) we can evaluate the strains within an element as:

$$\varepsilon^{(m)}(x,y,z) = B^{(m)}(x,y,z)\,\hat{U} \qquad \text{———} \quad (2)$$

where $B^{(m)}$ = Strain-Displacement Matrix that relates displacement to strains. obtained by appropriately differentiating the terms in $H^{(m)}$ matrix and combining appropriate terms.

The stresses within the finite element "m" can be expressed in terms of strains and initial stresses as

$$\tau^{(m)} = C^{(m)} \epsilon^{(m)} + \tau'^{(m)}$$

where $\tau'^{(m)} =$ element initial stresses.

Now we can write the virtual work statement (Eq. A) for the discretized 3-D body as:

$$\sum_m \int_{V^{(m)}} \bar{\epsilon}^{(m)T} \tau^{(m)} \, dV^{(m)} = \sum_m \int_{V^{(m)}} \bar{u}^{(m)} f^{B(m)} \, dV^{(m)}$$

$$+ \sum_m \int_{S_1^{m}, S_2^{(m)} \dots S_q^{(m)}} \bar{u}^{S(m)} f^{S(m)} \, dS^{(m)} \qquad \text{----} (B)$$

$$+ \sum_i \bar{u}^{iT} R_C^i$$

where $\bar{\epsilon}^{(m)}$ are element virtual strains and $\bar{u}^{(m)}$, $\bar{u}^{S(m)}$, and $\bar{u}^i$ are element virtual displacements.

m denotes the element number which varies from $m = 1, 2, \dots K$ for K number of elements

$S_1^{(m)} \dots S_q^{(m)}$ denotes the q element surfaces that are part of the Body Surface S_f on which surface tractions are prescribed.

General Derivation of Finite Element Equilibrium Equations

The virtual displacement field and strains can be expressed as:

$$\boxed{\begin{aligned} \bar{u}^{(m)}(x,y,z) &= H^{(m)}(x,y,z)\,\bar{\hat{U}} \\ \bar{\varepsilon}^{(m)}(x,y,z) &= B^{(m)}(x,y,z)\,\bar{\hat{U}} \end{aligned}} \quad \text{——— (C)}$$

where $H^{(m)}$ and $B^{(m)}$ are element Shape Function Matrix and Strain-Displacement Matrix respectively.
Substituting Eqns. (C) in Virtual Work Expression (B) we have the virtual Work Statement in terms of Nodal displacements as:

$$\boxed{\begin{aligned} \bar{\hat{U}}^T\left[\sum_m \int_{V^{(m)}} B^{(m)T} C^{(m)} B^{(m)}\, dV^{(m)}\right]\hat{U} = \bar{\hat{U}}^T\Bigg[&\left\{\sum_m \int_{V^{(m)}} H^{(m)T} f^{B(m)}\, dV^{m}\right\} \\ &+ \left\{\sum_m \int_{S_1^m \ldots S_q^{(m)}} H^{S(m)T} f^{S(m)}\, dS^{m}\right\} \\ &- \left\{\sum_m \int_{V^{(m)}} B^{(m)T} \tau^{I(m)}\, dV^{(m)}\right\} + R_c\Bigg] \end{aligned}} \quad \text{------- (D)}$$

In the above Virtual Work Statement $\hat{U}$ are the real displacements to be determined and $\bar{\hat{U}}$ are the virtual displacements imposed.

General Derivation of Finite Element Equilibrium Equations

In the previous equation Ⓓ $\gamma^{I(m)}$ are element Initial Strains that have been moved to the RHS of the eqn Ⓑ. R_c is the vector of concentrated Nodal Forces that are applied to the nodal pts of the discretized 3-D system.

vectors $\bar{U}$ and $\hat{U}$ are independent of the element m and the summation process and are therefore taken out of the sommation process.

To obtain the equilibrium equations for the system and to solve for the unknown nodal displacements we invoke the virtual displacements theorem by imposing unit virtual displacements in turn at all nodal displacement degrees of freedom.. In this way we have: $\bar{U}^T = I$ = Identity Matrix

With this substitution the virtual Work based Equation Ⓒ yields the Equilibrium Equation in the Famillar Form

$$\boxed{KU = R} \qquad \text{———} \quad (E)$$

where,

$$\boxed{R = R_B + R_S - R_I + R_C} \quad \text{———} \quad (F)$$

General Derivation of Finite Element Equilibrium Equations

In the previous Structure Equilibrium Equations (E)

The matrix K = Stiffness matrix of element assemblage

$$K = \sum_m \int_{V^{(m)}} \underbrace{B^{(m)T} C^{(m)} B^{(m)} \, dV^{(m)}}_{= K^{(m)} \text{ Element Stiffness Matrix.}} \quad \text{— (G)}$$

* In the above equation the assembly of the structural stiffness matrix from element stiffness matrices is carried out as per procedure of the <u>direct Stiffness Analysis</u> method. which is implied in the $\sum_m$ operand.

The Load Vector R contains the following components

R_B = Equivalent Nodal Load Vector corresponding to the element body forces

$$R_B = \sum_m \underbrace{\int_{V^{(m)}} H^{(m)T} f^{B(m)} \, dV^{m}}_{R_B^{m} = \text{ Element Body Forces Nodal load Vector}} \quad \text{— (H)}$$

Derivation of Finite Element Equilibrium Equations

R_S = Equivalent Nodal Load Vector corresponding to the element surface traction Forces

$$\boxed{R_S = \sum_m \underbrace{\int_{S_1^{m} \dots S_q^{(m)}} H^{S(m)T} f^{S(m)} \, dS^{(m)}}_{R_S^{(m)} = \text{Element Surface traction Nodal Load Vector}}} \quad \text{— (I)}$$

R_I = Equivalent Nodal Load Vector corresponding to Element Initial stresses

$$\boxed{R_I = \sum_m \underbrace{\int_{V^{(m)}} B^{(m)T} \tau^{I(m)} \, dV^{(m)}}_{R_I^{(m)} = \text{Element Initial Stresses Nodal Load Vector.}}} \quad \text{— (J)}$$

The Total Nodal Force Vector R contains also R_C, which is the nodal load vector of concentrated forces acting on the nodal pts.

$$\boxed{R_C = \text{Nodal Load Vector of Concentrated Forces directly applied/acting on the body.}} \quad \text{— (K)}$$

Derivation of Finite Element Equilibrium Equations

The form of Equilibrium Equations in Finite Element Analysis of Dynamics Problems is:

$$\boxed{M\ddot{U} + KU = R} \quad \text{——— (M)}$$

where R and U are time dependent quantities

and M = Equivalent Nodal Mass Matrix of the structure

$$M = \sum_m \int_{V^{(m)}} \rho^{(m)} H^{(m)T} H^{(m)} \, dV^{(m)}$$

General Derivation of Finite Element Equilibrium Equations

The virtual work statement in case of a dynamic problem can be written as follows:

$$\underbrace{\int_V \bar{U}^T \rho \ddot{u}\, dv}_{\text{work by Inertia Forces}} + \underbrace{\int_V \bar{U}^T c \dot{u}\, dv}_{\text{Work by Damping Forces}} + \underbrace{\int_V \bar{\varepsilon}^T \tau\, dv}_{\text{Work by internal Stresses or Strain Energy}} \quad \text{— (L)}$$

$$= \underbrace{\int \bar{U}^T f^B dv + \int_{S_f} \bar{U}^{S_f T} f^{S_f} ds + \sum_i \bar{U}^{iT} R_c^i}_{\text{Work done by body forces Surface tractions and applied Point Loads}}$$

with the substitutions that:

$$u^{(m)}(x,y,z) = H^{(m)}(x,y,z)\, \hat{U}$$
$$\dot{u}^{(m)} = H^{(m)}\, \hat{\dot{U}}$$
$$\ddot{u}^{(m)} = H^{(m)}\, \hat{\ddot{U}}$$

We can write the virtual work statement in terms of Structure Nodal displacements as:

$$\bar{U}^T \Big[\Big\{ \sum_m \int_{V^{(m)}} H^{(m)T} \rho H^{(m)} \Big\} \ddot{U} + \Big\{ \sum_m \int_{V^{(m)}} H^{(m)T} c\, H^{(m)} \Big\} \dot{U} \quad \text{— (M)}$$

$$+ \Big\{ \sum_m \int_{V^{(m)}} B^{(m)T} C^{(m)} B^{(m)} dv^m \Big\} \hat{U} = R$$

General Derivation of Finite Element Equilibrium Equations

Eqn (M) yields the equilibrium equations for a 3-D solid for dynamic conditions when $\bar{\hat{U}}^T = I$ identity matrix. The equilibrium equations are of the form:

$$\boxed{M\ddot{U} + C\dot{X} + KX = R} \quad \text{—— (N)}$$

where

$$\boxed{\begin{aligned} M &= \text{Mass Matrix} = \sum_m \int_{V^{(m)}} H^{(m)T} \rho^{(m)} H^{(m)}\, dV \\ C &= \text{Damping Matrix} = \sum_m \int_{V^{(m)}} H^{(m)T} c^{(m)} H^{(m)}\, dV \\ K &= \text{Stiffness Matrix} = \sum_m \int_{V^{(m)}} B^{(m)T} C^{(m)} B^{(m)}\, dV \end{aligned}} \quad \text{—— (O)}$$