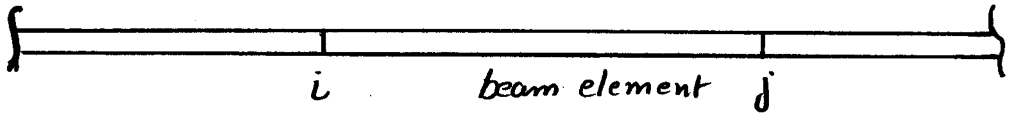
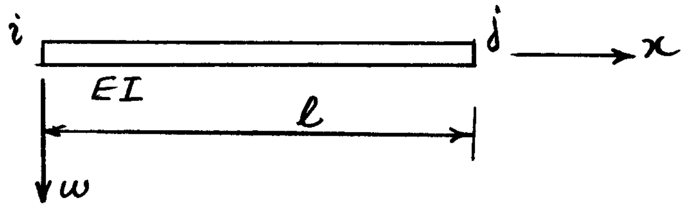
Development of Stiffness Matrix For a Beam Element

Consider the beam element shown below which is a part of a larger beam structure.



We first isolate the beam element of length l and ends lablelled i and j and assume that the beam is prismatic. The coordinate system adopted is also shown.



The trial function used for approximating beam deflections is taken as

$$w = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 \quad \text{———} ①$$

where α_i = undetermined constants

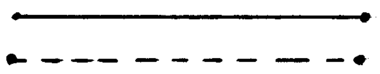
The above expression can be written in matrix form as:

$$w = \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \quad \text{———} ②$$

$$\equiv [N\alpha] \{\alpha\}$$

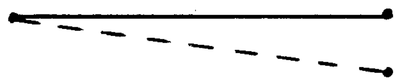
Development of Stiffness Matrix of a Beam Element

In the previous assumed displacement shape the term α_0 represent a rigid body translation, the term $\alpha_1 x$ represent a rigid body rotation, and $\alpha_2 x^2$ represents a constant curvature as shown below:



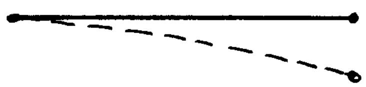
$w = \alpha_0$

Rigid Body translation



$w = \alpha_1 x$

Rigid Body Rotation



$w = \alpha_2 x^2$

Constant Curvature

We now relate the coefficients α_i to nodal values of deflection and rotation. This is necessary since when elements are connected together, certain constraints need to be imposed/met i.e the deflections and rotations for two beam elements meeting at a common node must be same to maintain deflection and slope compatibility.

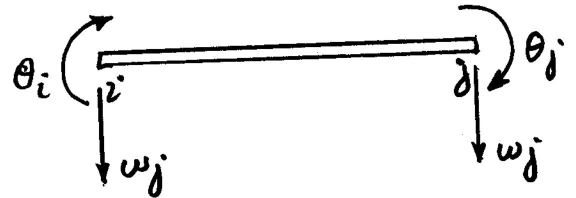
The deflection and slope at any pt. along the beam element would be given by the following relations according to the assumed deflection shape ①

$$\begin{Bmatrix} w(x) \\ \theta(x) \end{Bmatrix} = \begin{bmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \quad \text{——} \quad ③$$

Development of Stiffness Matrix for a Beam Element

For deflections and slopes at ends $i\ (x=0)$ and $j\ (x=\ell)$ of the beam element we can then write on the basis of equation ③:

$$\begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & \ell & \ell^2 & \ell^3 \\ 0 & 1 & 2\ell & 3\ell^2 \end{bmatrix} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \quad \text{———} \quad ④$$



The above equation can be written in abbreviated form as:

$$\{u\} = [A]\ \{\alpha\} \quad \text{———} \quad ⑤$$

Now the coefficients $\{\alpha\}$ can be determined in terms of member end displacements as:

$$\{\alpha\} = [A]^{-1}\{u\} \quad \text{———} \quad ⑥$$

$$[A]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/\ell^2 & -2/\ell & 3/\ell^2 & -1/\ell \\ 2/\ell^3 & 1/\ell^2 & -2/\ell^3 & 1/\ell^2 \end{bmatrix} \quad \text{———} \quad ⑦$$

Development of Stiffness Matrix for a Beam Element

The deflection w anywhere in the beam element can now be written in terms of the element end displacements as follows from eqs ② and ⑥

$$w(x) = [N\alpha]\,\{\alpha\} = \underbrace{[N\alpha]\,[A]^{-1}}_{N}\,\{u\}$$

$$w(x) = [N(x)]\,\{u\} \qquad \text{———} \quad ⑧$$

$$[N(x)] = [1 \quad x \quad x^2 \quad x^3] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/l^2 & -2/l & 3/l^2 & -1/l \\ 2/l^3 & 1/l^2 & -2/l^3 & 1/l^2 \end{bmatrix}$$

$$[Nx] = \left[\left(1 - 3\left(\frac{x}{l}\right)^2 + 2\left(\frac{x}{l}\right)^3\right),\; l\left(\frac{x}{l} - 2\left(\frac{x}{l}\right)^2 + \left(\frac{x}{l}\right)^3\right), \right. \qquad \text{———} \quad ⑨$$

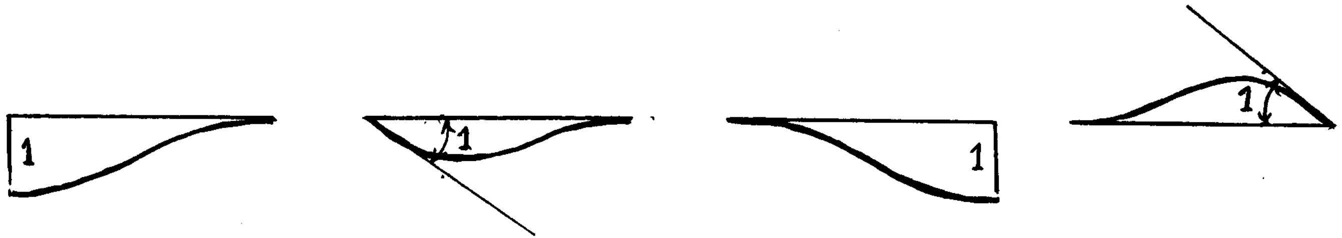
$$\left. 3\left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3,\; l\left(-\left(\frac{x}{l}\right)^2 + \left(\frac{x}{l}\right)^3\right)\right]$$

$[N(x)]$ = Shape Function Matrix referred to nodal displacements and rotations.

The shape functions contained in Eqn ⑨ are called "Hermitian Polynomials" since they interpolate using both the function itself (displacement) and its derivatives (rotations)

Development of Stiffness Matrix for a Beam Element

The plots of Hermitian polynomial shape functions are shown below:



$N_1(x) \equiv w_i$ $\quad 1-3\left(\frac{x}{l}\right)^2+2\left(\frac{x}{l}\right)^3$

$N_2(x) \equiv \theta_i$ $\quad l\left(\frac{x}{l}-2\left(\frac{x}{l}\right)^2+\left(\frac{x}{l}\right)^3\right)$

$N_3(x) \equiv w_j$ $\quad 3\left(\frac{x}{l}\right)^2-2\left(\frac{x}{l}\right)^3$

$N_4(x) \equiv \theta_j$ $\quad l\left(-\left(\frac{x}{l}\right)^2+\left(\frac{x}{l}\right)^3\right)$

We Now use the principle of virtual work to form the stiffness matrix for the beam element. The virtual work statement for the beam element can be written as follows

$$\int_0^l \delta K^T M \, dx = \int_0^l \delta w^T q(x) \, dx$$

Internal Virtual Work $\qquad$ External Virtual Work

δK = Virtual Imposed Curvature = $\delta\left(-\frac{d^2u}{dx^2}\right)$

δw = " " displacement = δw

M = Actual Moment = $EI\ K = -EI\frac{d^2u}{dx^2}$

$$K = -\frac{d^2u}{dx^2} = -\frac{d^2N_\alpha}{dx^2}\{\alpha\} = -\frac{d^2N_x}{dx^2}\{u\}$$

$$K = \underbrace{\begin{bmatrix} 0 & 0 & -2 & -6x \end{bmatrix}}_{[B_\alpha]} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix}$$

Development of Stiffness Matrix for a Beam Element

$$K = \underbrace{\left[\frac{6x}{\ell^2} - \frac{12x}{\ell^3} , \; \frac{4}{\ell} - \frac{6x}{\ell} , \; -\frac{6}{\ell^2} + \frac{12x}{\ell^3} , \; \frac{2}{\ell} - \frac{6x}{\ell} \right]}_{[B_u]} \begin{Bmatrix} w_i \\ \theta_i \\ w_j \\ \theta_j \end{Bmatrix}$$

From Principle of virtual work we have

$$\delta W_{int} = \text{Internal virtual Work}$$

$$= \int_0^\ell \delta \kappa^T M \, dx \quad = \{\delta \bar{u}\}^T \underbrace{\int_0^\ell [B]^T \; EI \; [B] \, dx}_{\text{Element Stiffness Matrix}} \{u\}$$

$$\begin{array}{l}\text{Element Stiffness} \\ \text{Matrix} \\ [K]\end{array} = EI \int_0^\ell [B]^T [B] \, dx \, ..$$

$$[K] = EI \int_0^\ell \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{Bmatrix} \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix} dx$$

$$\boxed{[K] = EI \int_0^\ell \begin{bmatrix} B_1^2 & B_1 B_2 & B_1 B_3 & B_1 B_4 \\ & B_2^2 & B_2 B_3 & B_2 B_4 \\ & \text{Symm.} & B_3^2 & B_3 B_4 \\ & & & B_4^2 \end{bmatrix} dx} \quad \text{—} (10)$$

Derivation of Stiffness Matrix for a Beam Element

$$B^T EI B = \int_0^1 \begin{bmatrix} \frac{36}{\ell^4}(1-2\xi)^2 & \frac{12}{\ell^3}(1-2\xi)(2-3\xi) & -\frac{36}{\ell^4}(1-2\xi)^2 & \frac{12}{\ell^3}(1-2\xi)(1-3\xi) \\ \ddots & \frac{4}{\ell^2}(2-3\xi)^2 & -\frac{12}{\ell^3}(1-2\xi)(1-3\xi) & \frac{4}{\ell^2}(2-3\xi)(1-3\xi) \\ & \ddots & \frac{36}{\ell^4}(1-2\xi)^2 & -\frac{12}{\ell^3}(1-2\xi)(1-3\xi) \\ & & \ddots & \frac{4}{\ell^2}(1-3\xi)^2 \end{bmatrix}$$

where $\xi = \frac{x}{\ell}$ ——— ⑪

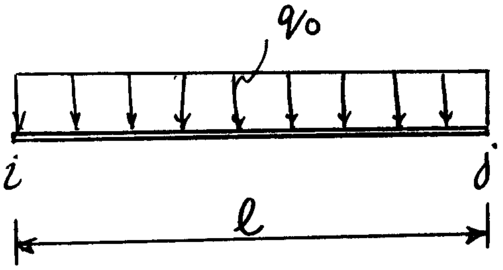
After carrying out the integration we have the stiffness matrix for the beam element as:

$$\boxed{[K] = \frac{EI}{\ell^3} \begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ \ddots & 4\ell^2 & -6\ell & 2\ell^2 \\ & \ddots & 12 & -6\ell \\ \text{Symm.} & & \ddots & 4\ell^2 \end{bmatrix}}$$ ——— ⑫

Stiffness Matrix for Beam Element.

Derivation of Equivalent Nodal Forces For a Beam Element

Case of Uniformly Distributed Load



External Virtual Work under Virtual Displacement $= \displaystyle\int_0^l \delta^T w \; q(x)\; dx$

$$= \delta \bar{U}^T \int_0^l [N_x]^T q_0 \; dx.$$

$$= \delta \bar{U}^T \int_0^l \begin{Bmatrix} 1 - 3\left(\frac{x}{l}\right)^2 + 2\left(\frac{x}{l}\right)^3 \\ l\left(\frac{x}{l} - 2\left(\frac{x}{l}\right) + \left(\frac{x}{l}\right)^3\right) \\ 3\left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3 \\ l\left(-\left(\frac{x}{l}\right)^2 + \left(\frac{x}{l}\right)^3\right) \end{Bmatrix} q_0 \, dx$$

If $\xi = \frac{x}{l}$

Then $dx = d\xi . l$, $\displaystyle\int_0^l f(x)\, dx = l \int_0^1 f(\xi)\, d\xi$

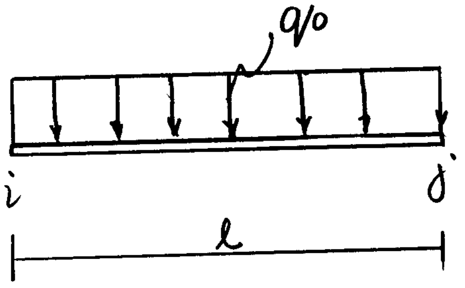
$$\delta W_{ext} = \delta \bar{U}^T . \; l \underbrace{\int_0^1 \begin{Bmatrix} 1 - 3\xi^2 + 2\xi^3 \\ l\left(\xi - 2\xi + \xi^3\right) \\ 3\xi^2 - 2\xi^3 \\ l\left(-\xi^2 + \xi^3\right) \end{Bmatrix} q_0 \, d\xi}_{\text{Equivalent Nodal Load Vector.}}$$

Equivalent Nodal Forces Vector for Beam Element

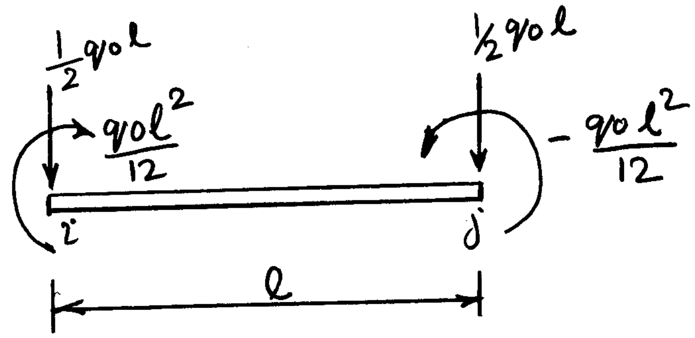
After carrying out the necessary integration we have for the "Equivalent Nodal Force Vector"

$$\boxed{\{P\} = q_0 l \begin{Bmatrix} 1/2 \\ l/12 \\ 1/2 \\ -l/12 \end{Bmatrix}} \quad \text{—— ⑬}$$

Equivalent Nodal Force Vector for UDL over Beam Element



Actual Loading



Equivalent Nodal Loads

Note:- Note that the Equivalent Nodal Loads are just the opposite of Fixed End Moments and Forces for the Beam Element.

TWO DIMENSIONAL ELASTICITY PROBLEMS

Many stress analysis problems can be approximated as two-dimensional, either plane stress or plane strain. A typical plane stress approximation would be a thin plate subjected to stresses in the plane perpendicular to the small dimension. The assumption would be that the direct stress in the "thickness" direction is zero, and the "inplane" stresses are constant through the thickness. A typical plain strain problem would be one in which the body is very large in one direction and because of symmetry conditions all deformation can be considered to take place in one plane, e.g., a two-dimensional "slice" of an embankment or dam.

For plane stress, the stress-strain relations (for isotropic case) are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1 - \nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

For plane strain, the same relations can be used if fictitious material properties E' and ν' are used, such that

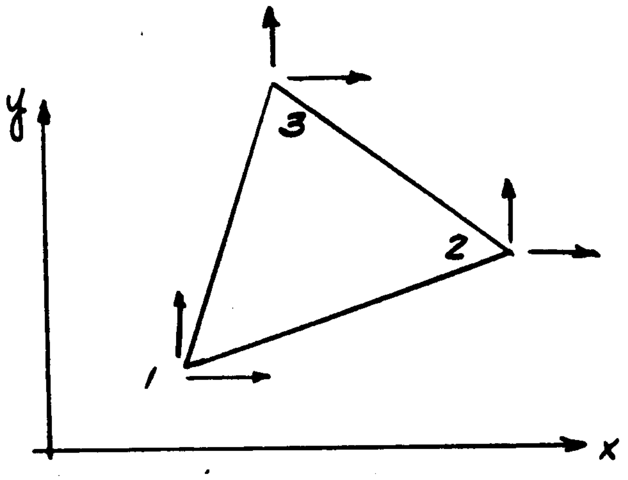
$$E' = \frac{E}{1-\nu^2} , \quad \nu' = \frac{\nu}{1-\nu}$$

In the following, only the plane stress case will be dealt with. The above relations can be used to transform to the plane strain case. For a beam element loaded at its ends by concentrated forces, it is possible to develop an exact stiffness matrix, (which was presented earlier). This is not possible for part of a continuum. The reason for this difference is that while for a beam (internally determinate) once the end forces are known what goes on between the end points can then be determined just from statics; this is not the case for a continuum, since it is internally indeterminate.

In other words, even if the edge forces acting or part of a continuum are known, the internal state of the continuum is not determined solely by statics. Thus, the problem of finding an exact stiffness matrix for a continuum is just as difficult as the original stress analysis problem.

We approach the problem in exactly the same way as we did the beam problem - however, we realize that exact results will very seldom be obtained. The simplest element for two dimensional problems is the constant strain triangle, (the triangular shape is more advantageous for handling irregular shapes than rectangular elements).

Constant Strain Triangle (CST)



Once we have chosen the three nodes, the number of degrees of freedom are fixed at six. (Displacements u, v at each node.)

$$u = \alpha_0 + \alpha_1 x + \alpha_2 y \qquad \text{or} \begin{Bmatrix} u \\ v \end{Bmatrix} = \underset{N_\alpha}{\begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix}} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix}$$

$$v = \alpha_3 + \alpha_4 x + \alpha_5 y$$

This is a complete polynomial of the 1st degree, and since the strain energy contains derivatives of 1st order, this should satisfy the "completeness" requirement. The terms α_0 , α_3 represent rigid body translations, the term $(\alpha_2 - \alpha_4)$ represents a rigid body rotation. α_1 represents a uniform strain in x-direction, α_5 represents a uniform strain in the y-direction, and $\alpha_2 + \alpha_4$ represents a uniform γ_{xy}.

The other requirement which should be met is that u,v be continuous between elements. Let us first transform from the $\{\alpha\}$ generalized coordinates to $\{u\}$.

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix}}_{[A]} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix}$$

is obtained by evaluating $[N_\alpha]$ at each node. (x_1,y_1) coordinates of node 1, etc.

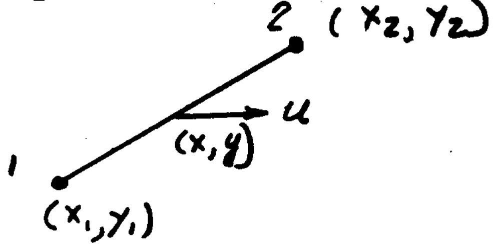
then

$$\begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{Bmatrix} = \frac{1}{2A} \underbrace{\begin{bmatrix} (x_2y_3-x_3y_2) & 0 & (x_3y_1-x_1y_3) & 0 & (x_1y_2-x_2y_1) & 0 \\ y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ x_{32} & 0 & x_{13} & 0 & x_{21} & 0 \\ 0 & (x_2y_3-x_3y_2) & 0 & (x_3y_1-x_1y_3) & 0 & (x_1y_2-x_2y_1) \\ 0 & y_{23} & 0 & y_{31} & 0 & y_{12} \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \end{bmatrix}}_{[A^{-1}]} \begin{bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{bmatrix}$$

A is the area of the triangle $= \frac{1}{2}(x_1y_{23} + x_2y_{31} + x_3y_{12})$

$y_{23} \equiv y_2 - y_3$ etc.

Let's now examine whether or not u, v are continuous between elements. On side 1-2, $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$ (eqn of bd'y)



The variation of u along side 1-2 is

$$u_{1-2} = \alpha_0 + \alpha_1 x + \alpha_2 [(\frac{y_2-y_1}{x_2-x_1})(x-x_1) + y_1]$$

$$= \frac{1}{2A} [(x_2y_3 - x_3y_2)u_1 + (x_3y_1 - x_1y_3)u_2 + (x_1y_2 - x_2y_1)u_3$$

$$+(y_{23} u_1 + y_{31} u_2 + y_{12} u_3)x$$

$$+(x_{32} u_1 + x_{13} u_2 + x_{21} u_3)(\frac{y_{21}}{x_{21}} (x-x_1) +y_1)]$$

$$= \frac{1}{2A} [(\quad)u_1 + (\quad)u_2$$

$$\underbrace{+(x_1y_2 - x_2y_1 + y_{12} x + x_{21}[\frac{y_{21}}{x_{21}} (x-x_1)+ y_1])u_3]}_{\cancel{x_1y_2} - \cancel{x_2y_1} - \cancel{y_{21}x_1} + \cancel{x_{21}y_1} + x(\cancel{y_{12} + y_{21}})}$$

$$u_{1-2} = \frac{1}{2A} [(\quad) u_1 + (\quad)u_2]$$

i.e. u along side 1-2 depends only on u_1 and u_2 (in fact a linear variation along edge). A neighboring element of the same type will have a boundary displacement depending only on nodes on the common boundary also. If we make the nodal displacements of adjoining elements the same at common nodes, then th boundary displacements between nodes will be the same also; the element is conforming. Note that strains and therefore stresses are not continuous between (CST)elements. We now continue with the development of the element stiffness matrix.

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

$\varepsilon_x = \frac{\partial}{\partial x}(\alpha_0 + \alpha_1 X + \alpha_2 y) = \alpha_1$ etc. In matrix form

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} [N_\alpha]\{\alpha\}$$

$$= \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}}_{[B_\alpha]} \{\alpha\}$$

Note that $[B_\alpha]$ does not depend on (x, y) i.e. the strains are constant throughout the element. Hence, the name "Constant Strain Triangle" (CST). In terms of nodal displacements,

$$\{\varepsilon\} = \underbrace{[B_\alpha]\ [A^{-1}]}_{[B]}\ \{u\}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \underbrace{\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}}_{[B]} \begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \\ U_3 \\ V_3 \end{Bmatrix}$$

The element stresses for plane stress are

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \underbrace{\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}}_{[D]} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

or $\{\sigma\} = [D]\ \{\varepsilon\}$

thus $\{\sigma\} = [D]\ [B]\ \{u\}$. This is sometines called the "stress" matrix

The principle of virtual work is now applied by imposing a virtual displacement $\{\delta u\}$ at the nodes. (This can be any one or combination of $\delta u_1, \delta v_1, \delta u_2, \ldots, \delta v_3$)

$$\delta W_{int} = - \int_V \{\delta\varepsilon\}^T\{\sigma\}dV \quad \left(= - \int_V (\delta\varepsilon_x\sigma_x + \delta\varepsilon_y\sigma_y + \delta\gamma_{xy}\tau_{xy})dV\right)$$

$$\{\delta\varepsilon\} = [B]\{\delta u\} \quad , \quad \{\delta\varepsilon\}^T = \{\delta u\}^T[B]^T$$

$$\delta W_{int} = - \{\delta u\}^T \underbrace{\left[\int_{Area} t[B]^T[D][B]dxdy \right]_{6x6}}_{[k]} \{u\}$$

t = thickness in z thickness

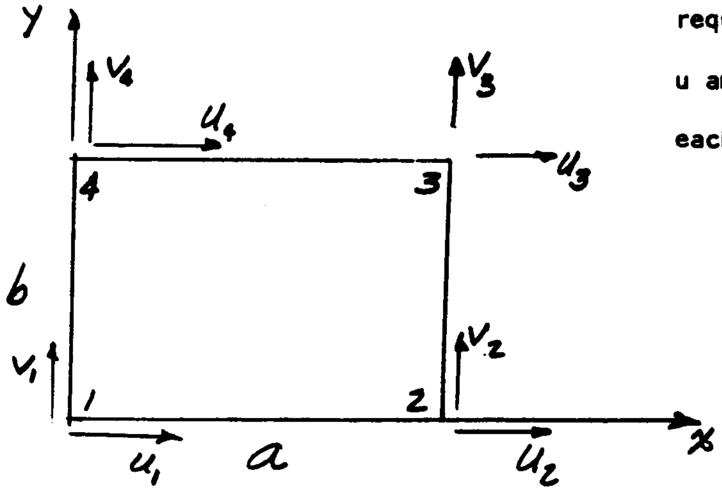
In this case, $[B]$ is independent of (x, y) so $[k] = tA\ [B]^T[D][B]$. Each virtual displacement generates one row of $[k]$, or the "equilibrium" equation corresponding to that virtual displacement.

Stiffness Matrix for CST Element

$$[k] = \frac{Et}{4A(1-\nu^2)} \left[\begin{array}{cccccc|c}
u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & \\
\hline
Y_{23}^2 + \left(\frac{1-\nu}{2}\right) X_{32}^2 & & & & & & u_1 \\
\left(\frac{1+\nu}{2}\right) X_{32} Y_{23} & X_{32}^2 + \left(\frac{1-\nu}{2}\right) Y_{23}^2 & & \text{Symmetric} & & & v_1 \\
Y_{31} Y_{23} + \left(\frac{1-\nu}{2}\right) X_{13} X_{32} & \nu X_{32} Y_{31} + \left(\frac{1-\nu}{2}\right) X_{13} Y_{23} & Y_{31}^2 + \left(\frac{1-\nu}{2}\right) X_{13}^2 & & & & u_2 \\
\nu X_{13} Y_{23} + \left(\frac{1-\nu}{2}\right) X_{32} Y_{31} & X_{13} X_{32} + \left(\frac{1-\nu}{2}\right) Y_{23} Y_{31} & \left(\frac{1+\nu}{2}\right) X_{13} Y_{31} & X_{13}^2 + \left(\frac{1-\nu}{2}\right) Y_{31}^2 & & & v_2 \\
Y_{12} Y_{23} + \left(\frac{1-\nu}{2}\right) X_{21} X_{32} & \nu X_{32} Y_{12} + \left(\frac{1-\nu}{2}\right) X_{21} Y_{23} & Y_{12} Y_{31} + \left(\frac{1-\nu}{2}\right) X_{13} X_{21} & \nu X_{13} Y_{12} + \left(\frac{1-\nu}{2}\right) X_{21} Y_{31} & Y_{12}^2 + \left(\frac{1-\nu}{2}\right) X_{21}^2 & & u_3 \\
\nu X_{21} Y_{23} + \left(\frac{1-\nu}{2}\right) X_{32} Y_{12} & X_{21} X_{32} + \left(\frac{1-\nu}{2}\right) Y_{12} Y_{23} & \nu X_{21} Y_{31} + \left(\frac{1-\nu}{2}\right) X_{13} Y_{12} & X_{13} X_{21} + \left(\frac{1-\nu}{2}\right) Y_{12} Y_{31} & \left(\frac{1+\nu}{2}\right) X_{21} Y_{12} & X_{21}^2 + \left(\frac{1-\nu}{2}\right) Y_{12}^2 & v_3
\end{array} \right]$$

$X_{12} \equiv X_1 - X_2$, etc.

Rectangular Element (P5R)



The rectangular element with 4 nodes requires a polynomial expansion for u and v with 4 undetermined constants each.

$$\begin{aligned} u &= \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 xy \\ v &= \alpha_4 + \alpha_5 x + \alpha_6 y + \alpha_7 xy \end{aligned} \qquad \text{or} \qquad \begin{Bmatrix} u \\ v \end{Bmatrix} = [N_\alpha]\{\alpha\}$$

Note that one quadratic term, xy, is chosen from the three possibilities x^2, xy, y^2. This particular choice is made to preserve symmetry and to insure that u, v will be continuous between elements. As with the CST element, all rigid body modes and constant strain states are included (in terms $\alpha_0 + \alpha_1 x + \alpha_2 y$ and $\alpha_4 + \alpha_5 x + \alpha_6 y$).

Now transform from $\{\alpha\}$ to $\{u\}$. To simplify things, we will use the order of listing $(u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4)$ and later transform to the order we want in the final result, i.e. $(u_1, v_1, u_2, v_2 \ldots)$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix} = \underbrace{\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & & & & \\ 1 & a & 0 & 0 & & [0]_{4\times4} & & \\ 1 & a & b & ab & & & & \\ 1 & 0 & b & 0 & & & & \\ \hline & & & & 1 & 0 & 0 & 0 \\ & & & & 1 & a & 0 & 0 \\ & [0]_{4\times4} & & & 1 & a & b & ab \\ & & & & 1 & 0 & b & 0 \end{array}\right]}_{[A]} \begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{Bmatrix}$$

Only need to
to invert 4x4
when this order
of listing used.

$$\begin{Bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{Bmatrix} = \underbrace{\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & & & & \\ -\frac{1}{a} & \frac{1}{a} & 0 & 0 & & [0]_{4x4} & & \\ -\frac{1}{b} & 0 & 0 & \frac{1}{b} & & & & \\ \frac{1}{ab} & -\frac{1}{ab} & \frac{1}{ab} & \frac{-1}{ab} & & & & \\ \hline & & & & 1 & 0 & 0 & 0 \\ & & & & \frac{-1}{a} & \frac{1}{a} & 0 & 0 \\ & [0]_{4x4} & & & \frac{-1}{b} & 0 & 0 & \frac{1}{b} \\ & & & & \frac{1}{ab} & \frac{-1}{ab} & \frac{1}{ab} & \frac{-1}{ab} \end{array}\right]}_{[A^{-1}]} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix}$$

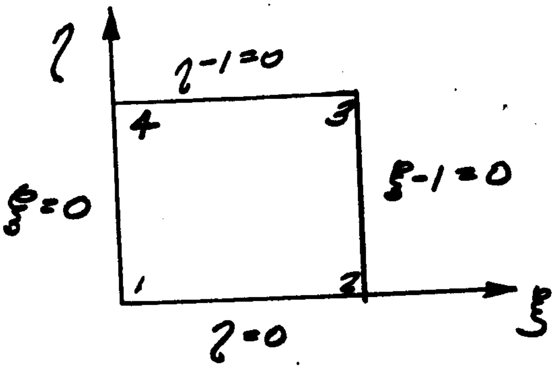
Thus, written directly in terms of nodal unknowns, the expression for U is

$$u = (1 - \frac{x}{a} - \frac{y}{b} + \frac{xy}{ab}) u_1 + (\frac{x}{a} - \frac{xy}{ab}) u_2 + \frac{xy}{ab} u_3 + (\frac{y}{b} - \frac{xy}{ab}) u_4$$

In terms of dimensionless coordinates, $\xi = x/a$, $\eta = y/b$

$$u = (1 - \zeta)(1 - \eta) u_1 + \zeta(1 - \eta) u_2 + \zeta\eta\ u_3 + \eta(1 - \zeta) u_4$$

and a similar expression for v.



This expression could have been written down directly by observing that each shape function must give a unit nodal value for the particular nodal displacement under consideration and zero nodal values for all others. For node 1, we want a shape function which will be zero at nodes 2, 3, and 4. This can be obtained by taking the product of equations of the sides 2-3 and 3-4, i.e. $(1-\xi)(1-\eta)$. At $\zeta = 0$, $\eta = 0$ (node 1) this expression gives a unit value, thus no normalizing factor needs to be included. With the displacements written directly in terms of nodal unknowns, it is easy to verify that u, v are continuous between elements, i.e., $u_{2-3} = (1 - \eta)u_2 + \eta u_3$. u_{2-3} depends only on u_2, u_3.

Development of Stiffness Matrix

$$\begin{matrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{matrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x \\ 0 & 0 & 1 & x & 0 & 1 & 0 & y \end{bmatrix}}_{[B_\alpha]} \{\alpha\} = [B_\alpha]\,[A^{-1}]\,\{u\}$$

Note that now [B] depends on x and y, thus strains, and therefore stresses depend on location within the element.

$$\begin{matrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{matrix} \quad \underbrace{\left[\begin{array}{c|c|c|c|c|c|c|c} -\frac{1}{a}(1-\frac{y}{b}) & \frac{1}{a}(1-\frac{y}{b}) & \frac{1}{a}\,\frac{y}{b} & -\frac{1}{a}\,\frac{y}{b} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{b}(1-\frac{x}{a}) & -\frac{1}{b}\,\frac{x}{a} & \frac{1}{b}\,\frac{x}{a} & \frac{1}{b}(1-\frac{x}{a}) \\ -\frac{1}{b}(1-\frac{x}{a}) & -\frac{1}{b}\,\frac{x}{a} & \frac{1}{b}\,\frac{x}{a} & \frac{1}{b}(1-\frac{x}{a}) & -\frac{1}{a}(1-\frac{y}{b}) & \frac{1}{a}(1-\frac{y}{b}) & \frac{1}{a}\,\frac{y}{b} & -\frac{1}{a}\,\frac{y}{b} \end{array}\right]}_{[B]} \begin{Bmatrix} u_1 \\ \vdots \\ u_4 \\ \vdots \\ v_1 \\ \vdots \\ v_4 \end{Bmatrix}$$

The stiffness matrix is evaluated, as before, by calculating the internal virtual work due to a nodal virtual displacement $\{\delta u\}$. (Actually eight independent virtual displacements,

$$[\ \delta u = (1 - \frac{x}{a} - \frac{y}{b} + \frac{xy}{ab})\ \delta u_1,\ \delta v = 0],\ \text{etc.})$$

$$\delta W_{int} = - \int_V \{\delta\varepsilon\}^T\{\sigma\}dV = - \int_{Area} t\{\delta\varepsilon\}^T\{\sigma\}dxdy$$

$$\text{Stresses } \{\sigma\} = [D]\{\varepsilon\} = [D][B_\alpha]\{\alpha\} = [D][B_\alpha][A^{-1}]\{u\}$$

$$= [D][B]\{u\}$$

$$\text{Virtual strains } \{\delta\varepsilon\} = [B_\alpha][A^{-1}]\{\delta u\} = [B]\{\delta u\}$$

$$\delta W_{int} = - \{\delta u\}^T \underbrace{\left[\int_{Area} t[B]^T[D][B]dxdy\right]}_{[k]}\{u\}$$

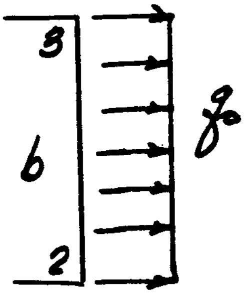
or $$[k] = [A^{-1}]^T \underbrace{\left[\int_{Area} t[B_\alpha]^T[D][B_\alpha]dxdy\right]}\ [A^{-1}]$$

This is essentially a stiffness matrix "in the $\{\alpha\}$ coordinate system" $= [k_\alpha]$

Once [k] has been determined, the rows and columns are rearranged to correspond to the order of listing (u_1, v_1, u_2, v_2, ... v_4). The stiffness matrix is given on the following page.

Equivalent Nodal Loads

The equivalent nodal loads corresponding to the actual distributed loading (if any) are computed by calcualting external virtual work. For example, suppose a uniformly distributed edge force acts on boundary 2-3 of the element.



$$\delta W_{ext} = \int_0^b \lceil \delta u \;\; \delta v \rfloor \Big|_{2-3} \begin{Bmatrix} q_o \\ o \end{Bmatrix} dy$$

Stiffness Matrix for PSR Element

$$
k] = \frac{Et}{12(1-\nu^2)}
\begin{array}{cccccccc|c}
u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 & \\
\hline
4\beta+2(1-\nu)\beta^{-1} & & & & & & & & u_1 \\
\frac{3}{2}(1+\nu) & 4\beta^{-1}+2(1-\nu)\beta & & & & & & & v_1 \\
-4\beta+(1-\nu)\beta^{-1} & \frac{3}{2}(1-3\nu) & 4\beta+2(1-\nu)\beta^{-1} & & & \text{Symmetric} & & & u_2 \\
-\frac{3}{2}(1-3\nu) & 2\beta^{-1}-2(1-\nu)\beta & -\frac{3}{2}(1+\nu) & 4\beta^{-1}+2(1-\nu)\beta & & & & & v_2 \\
-2\beta-(1-\nu)\beta^{-1} & -\frac{3}{2}(1+\nu) & 2\beta-2(1-\nu)\beta^{-1} & \frac{3}{2}(1-3\nu) & 4\beta+2(1-\nu)\beta^{-1} & & & & u_3 \\
-\frac{3}{2}(1+\nu) & -2\beta^{-1}-(1-\nu)\beta & -\frac{3}{2}(1-3\nu) & -4\beta^{-1}+(1-\nu)\beta & \frac{3}{2}(1+\nu) & 4\beta^{-1}+2(1-\nu)\beta & & & v_3 \\
2\beta-2(1-\nu)\beta^{-1} & -\frac{3}{2}(1-3\nu) & -2\beta-(1-\nu)\beta^{-1} & \frac{3}{2}(1+\nu) & -4\beta+(1-\nu)\beta^{-1} & \frac{3}{2}(1-3\nu) & 4\beta+2(1-\nu)\beta^{-1} & & u_4 \\
\frac{3}{2}(1-3\nu) & -4\beta^{-1}+(1-\nu)\beta & \frac{3}{2}(1+\nu) & -2\beta^{-1}-(1-\nu)\beta & -\frac{3}{2}(1-3\nu) & 2\beta^{-1}-2(1-\nu)\beta & -\frac{3}{2}(1+\nu) & 4\beta^{-1}+2(1-\nu)\beta & v_4
\end{array}
$$

$$\beta \equiv b/a$$

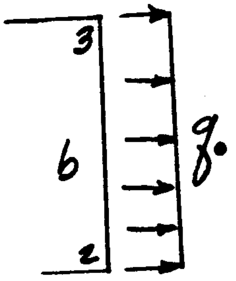
$$\begin{Bmatrix}\delta u\\ \delta v\end{Bmatrix} = \begin{bmatrix}N\end{bmatrix}\{u\}_{x=a}$$

$$\lceil \delta u \;\; \delta v \rfloor = \{\delta u\}^T [N]^T_{x-a}$$

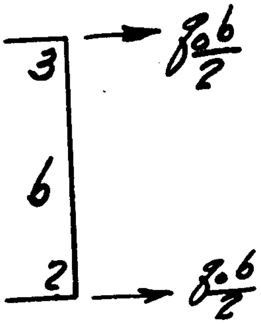
$$\delta W_{ext} = \{\delta u\}^T \underbrace{\int_0^b [N]^T_{x=a}\begin{Bmatrix}q_o\\ 0\end{Bmatrix} dy}_{\{p\}}$$

$$\{p\} = \int_0^b \left[\begin{array}{c|c} 0 & 0\\ 1-y/b & 0\\ y/b & 0\\ 0 & 0\\ \hline 0 & 0\\ 0 & 1-y/b\\ 0 & y/b\\ 0 & 0\end{array}\right]\begin{Bmatrix}q_o\\ 0\end{Bmatrix} dy = \int_0^b q_o \begin{Bmatrix}0\\ 1-y/b\\ y/b\\ 0\\ \hdashline 0\\ 0\\ 0\\ 0\end{Bmatrix} dy = \frac{q_0 b}{2}\begin{Bmatrix}0\\ 1\\ 1\\ 0\\ \hdashline 0\\ 0\\ 0\\ 0\end{Bmatrix}$$

Note this corresponds to order of listing (u_1, u_2, $u_3 \ldots v_4$)



Actual Loading



Equivalent Nodal Loads

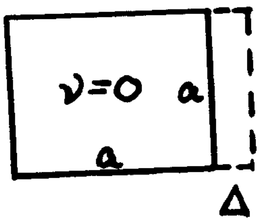
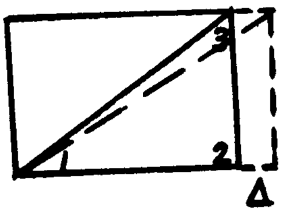
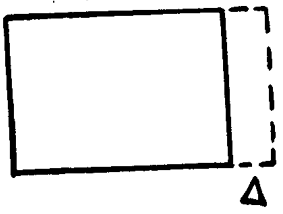
Note total horizontal resultant = $q_o b$ for both. Division of distributed loads to nodes depends on the assumed displacement variation.

$(\varepsilon_x = (1-\eta)(\frac{u_2-u_1}{a}) + \eta(\frac{u_3-u_4}{a}))$. Note also,

1) element shear stress at free end is zero, (actual parabolic)
2) shear stress varies linearly in x-dir (actual is constant)
3) bending moment at fixed edge = $\frac{1}{3}$ actual.

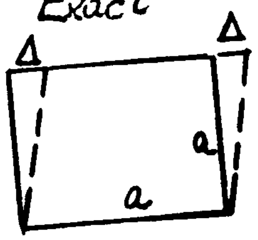
Comparison of CST and PSR elements

From the preceding example, it appears that the constant strain triangle (CST) may be significantly stiffer, and therefore inferior to the rectangular element (PSR). In the chapter on "Convergence of Finite Element Models", it was shown that a comparison of eigenvalues of two stiffness matrices could be used as a test of superiority of one over the other. The eigenvectors correspond to basic deformation patterns such as uniform extension, shear, pure bending. Therefore, to compare the CST and PSR elements we will subject them to nodal displacement patterns corresponding to uniform extension, shear, and pure bending and compare the strain energies. The element developing the smallest strain energy for a given nodal displacement pattern is the "softest" and therefore the best.

	Exact	CST	PSR
Extension	 $\varepsilon_x = \frac{\Delta}{a}$, $\sigma_x = \frac{E\Delta}{a}$ $U = \frac{1}{2}a^2 t(\frac{E\Delta}{a})(\frac{\Delta}{a})$	 $X_{32}=0$ $y_{23}=-a$ $X_{13}=-a$ $y_{31}=a$ $X_{21}=a$ $y_{12}=0$ $\varepsilon_x = \frac{\Delta}{a}$, $\varepsilon_y=0$, $\gamma_{xy}=0$	 $\varepsilon_x = \frac{\Delta}{a}$, $\varepsilon_y=0$, $\gamma_{xy}=0$
	$U = \frac{1}{2}Et\Delta^2$	$U = \frac{1}{2}Et\Delta^2$	$U = \frac{1}{2}Et\Delta^2$

Shear

Exact

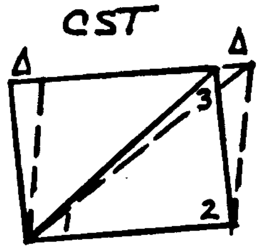


$\varepsilon_x = 0, \ \varepsilon_y = 0, \ \gamma_{xy} = \frac{\Delta}{a}$

$\tau_{xy} = \frac{E}{2}\frac{\Delta}{a}$

$$\boxed{U = \frac{1}{4}Et\Delta^2}$$

CST



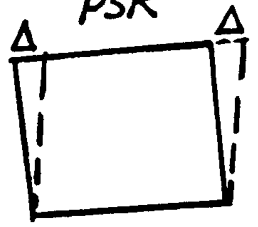
$\varepsilon_x = 0$

$\varepsilon_y = 0$

$\gamma_{xy} = \frac{\Delta}{a}$

$$\boxed{U = \frac{1}{4}Et\Delta^2}$$

PSR



$\varepsilon_x = 0$

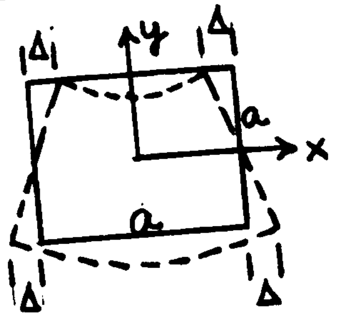
$\varepsilon_y = 0$

$\gamma_{xy} = \frac{\Delta}{a}$

$$\boxed{U = \frac{1}{4}Et\Delta^2}$$

Flexure

Exact



$u = -\frac{4xy}{a^2}\Delta$

$v = \frac{\Delta}{2}\left(\frac{4x^2}{a^2} - 1\right)$

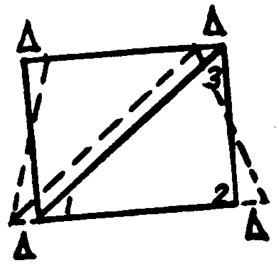
$\varepsilon_x = -\frac{4y}{a^2}\Delta \ , \ \varepsilon_y = 0$

$\boxed{\gamma_{xy} = 0}$

$U = \frac{1}{2}\int_{-\frac{a}{2}}^{a/2}\int_{-\frac{a}{2}}^{a/2} E\,\frac{16\Delta^2}{a^4}y^2 dy$

$$\boxed{U = \frac{2}{3}Et\Delta^2}$$

CST



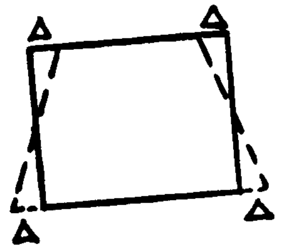
$\varepsilon_x = \frac{2\Delta}{a}, \ \varepsilon_y = 0$

$\boxed{\gamma_{xy} = -\frac{2\Delta}{a}}$

$$\boxed{U = 3Et\Delta^2}$$

4.5 × Exact

PSR



Since stresses & strains vary within the element, integrations can be avoided by computing nodal forces corresponding to the imposed displacements

$\{p\} = \frac{Et\Delta}{12}\begin{Bmatrix} -6 \\ = \\ 6 \\ = \\ -6 \\ = \\ 6 \\ = \end{Bmatrix}$ y-forces not needed

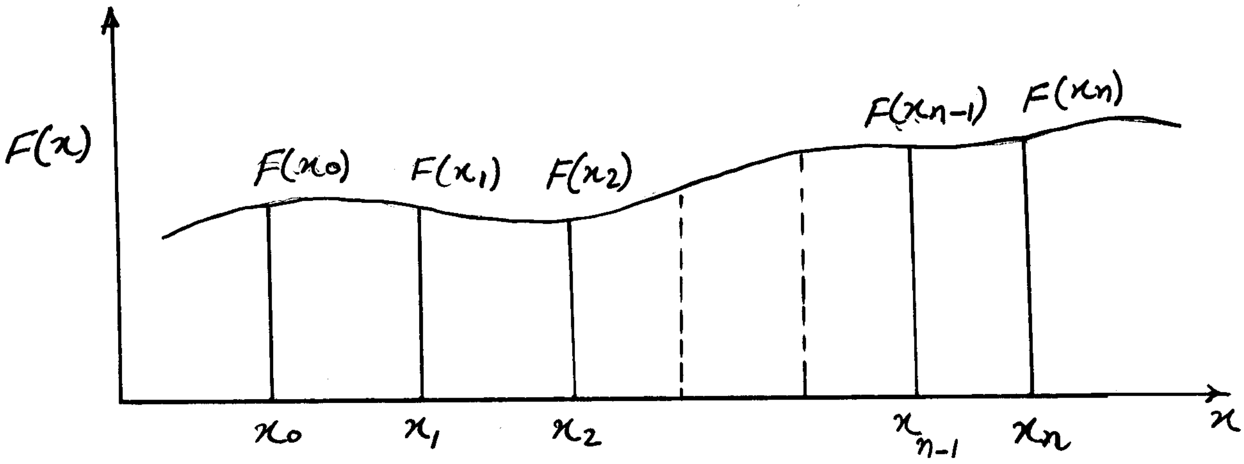
$$\boxed{U = Et\Delta^2}$$

1.5 × Exact

The CST is much too stiff in flexure. Both CST and PSR are stiffer than exact since they develop a shear strain under purely flexural loading. The PSR is much more flexible than CST, however.

Interpolation Formulae

Assume that a Function $F(x)$ has been evaluated at $(n+1)$ distinct points $x_0, x_1, x_2 \ldots\ldots x_n$ and it has values $F(x_0), F(x_1), F(x_2) \ldots\ldots F(x_n)$ at those points. The problem that can be posed is to find a polynomial $\phi(x)$ that passes through the Function values $F(x_0), F(x_1) \ldots\ldots F(x_n)$ at pts $x_0, x_1, x_2 \ldots\ldots x_n$.



There is a unique polynomial $\phi(x)$ of order n that satisfies the requirement of passing through the function values at sampling points $x_0, x_1 \ldots\ldots x_n$. The polynomial is given by general expression:

$$\phi(x) = a_0 + a_1 x + a_2 x^2 + \ldots\ldots a_n x^n$$

where, $a_0, a_1, a_2 \ldots\ldots a_n$ are $n+1$ coefficients that need to be determined.

Interpolation Formulae

Using the condition that the function $\phi(x)$ needs to pass through $F(x_0), F(x_1), ---- F(x_n)$ ordinates at $x_0, x_1, --- x_n$ locations, we can write:

$$\phi(x_0) = F(x_0) = a_1 + a_2 x_0 + a_3 x_0^2 + ---- a_n x_0^n$$
$$\phi(x_1) = F(x_1) = a_1 + a_2 x_1 + a_3 x_1^2 + ---- a_n x_1^n$$
$$\vdots \quad = \quad \vdots \qquad\qquad \vdots$$
$$\phi(x_n) = F(x_n) = a_1 + a_2 x_n + a_3 x_n^2 + ---- a_n x_n^n$$

Or in Matrix Form:

$$\underbrace{\begin{Bmatrix} F_0 \\ F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}}_{F} = \underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & ---- & x_n^n \\ 1 & x_1 & x_1^2 & ---- & x_1^n \\ 1 & x_2 & x_2^2 & ---- & x_2^n \\ \vdots & \vdots & \vdots & ---- & \vdots \\ 1 & x_n & x_n^2 & ---- & x_n^n \end{bmatrix}}_{V} \underbrace{\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{Bmatrix}}_{a}$$

$$\{F\} = [V]\{a\} \qquad \text{——— ①}$$

The matrix $[V]$ above is called the "Vandermode Matrix"

The solution for polynomial coefficients is then:

$$\{a\} = [V]^{-1}\{F\} \qquad \text{——— ②}$$

The solution is possible since $\{F\}$ has real values at $n+1$ distinct pts, therefore V^{-1} exists and we can determine a unique polynomial that satisfies the requirement of passing through $F_0, F_1, --- F_n$.

Interpolation Formulae

The process of interpolation described previously is cumbersome. An easier interpolation technique is the method proposed by Lagrange and is called Langrangian interpolation in which the interpolating polynomial is written as

$$F(x) \simeq \phi(x) = \sum_{i=0}^{n} l_i(x)\, F(x_i)$$

where $l_i(x)$ are interpolation shape functions and $F(x_i)$ are values of the Function $F(x)$ to be interpolated ie $F(x_0), F(x_1) ---- F(x_n)$

The interpolation shape functions are of the form

$$l_0(x) = \frac{(x_1-x)(x_2-x)----(x_n-x)}{(x_1-x_0)(x_2-x_0)----(x_n-x_0)}$$

$$l_1(x) = \frac{(x_0-x)(x_2-x)---(x_n-x)}{(x_0-x_1)(x_2-x_1)---(x_n-x_1)} \;/$$

$$\vdots$$

$$l_n(x) = \frac{(x_0-x)(x_1-x)---(x_{n-1}-x)}{(x_0-x_n)(x_1-x_n)----(x_{n-1}-x_n)}$$

$$\boxed{l_K(x_K) = \frac{(x_0-x)(x_1-x)--[x_K-x]---(x_n-x)}{(x_0-x_K)(x_1-x_K)--[x_K-x_K]---(x_n-x_K)}}$$

in which the bracketed term $\frac{[x_K-x]}{[x_K-x_K]}$ is omitted for Kth interpolation function.

Interpolation Formulae
1 Lagrangian interpolation.

An important property of Lagrangian interpolation shape functions $l_i(x)$ is that it has a value equal to "1" when evaluated at x_i and zero at at all other sampling locations i-e

$$l_i(x)\Big|_{x=x_i} = 1 \qquad \text{or} \qquad li(x_j) = \delta_{ij} = \begin{matrix} 1 \text{ for } i=j \\ 0 \text{ for } i\neq j \end{matrix}$$

and $$l_i(x)\Big|_{x\neq x_i} = 0$$

We will make use of Lagrangian interpolation when we derive shape functions for basic elements and isoparametric elements.

Introduction to Natural Coordinates

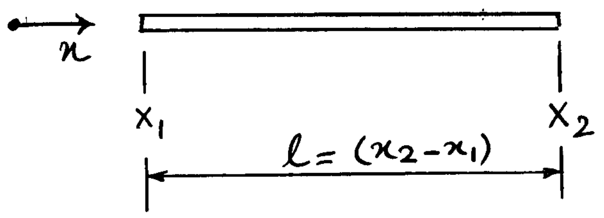
A "local Coordinate System" is a coordinate system that is defined for a particular element and not necessarily for the entire body or structures; the coordinate system for the entire body or structure is called the "Global Coordinate System".

A "Natural Coordinate System" is a local coordinate system which permits identification/referencing of a point within an element in terms of of dimensionless numbers whose magnitude never exceeds unity. (1)

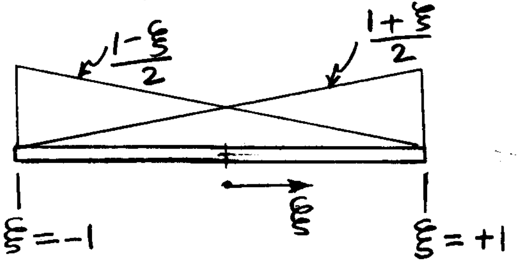
Natural Coordinates In One Dimension

Consider the bar element shown below in global coordinate system X and Natural Coordinate system ξ.

Global Coord System



Natural Coordinate System.



It is possible to establish a correspondence/Mapping between Global Coordinate system and the Natural coordinate system such that each pt. along the beam is uniquely mapped onto the beam element in the natural coordinate system and vice versa.

It is possible in this case to write an expression for ξ coordinate in terms of X coords

$$\xi = \frac{2x - (x_2+x_1)}{(x_2+x_1)}$$

$$\xi = +1 \quad \text{for } x = x_2$$
$$\xi = -1 \quad \text{for } x = x_1$$

Hence we have expressed the global coordinates X in terms of Natural coordinates ξ

* It is possible to write an expression for any pt x in Global Coordinate system in terms of shape functions defined in natural coordinate systems and Nodal coordinates X_1, X_2 of the Beam in Global system.

Using the Shape Functions shown in the Figure we can write

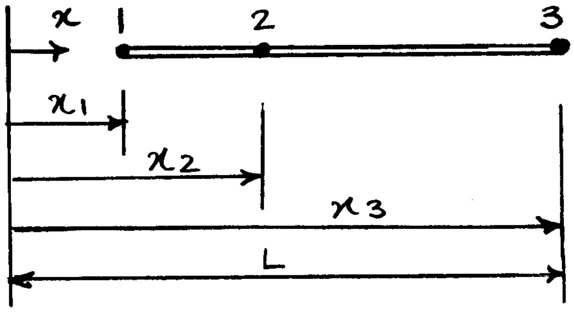
$$\boxed{\begin{aligned} x &= \left(\frac{1-\xi}{2}\right)x_1 + \left(\frac{1+\xi}{2}\right)x_2 \\ \text{or}\quad x &= \left[\frac{1-\xi}{2}, \ \frac{1+\xi}{2}\right]\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} \\ \text{or}\quad x &= \sum_{i=1}^{2} N_i(\xi)\, X_i \end{aligned}}$$

** Thus we have established mapping between natural Coordinate system and Global Coordinate system by Shape Functions in natural Coordinate System. This concept is extendable to 2-D elements and 3-D Elements as well.

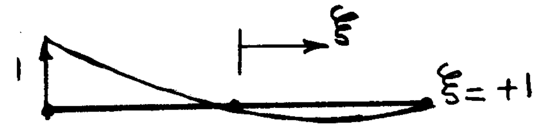
Natural Coordinates & Shape functions

Next we develop shape functions for a 3-Node bar element and demonstrate the usage of shape functions in natural coordinates to develop relation between natural Coordinates and Global Coordinates

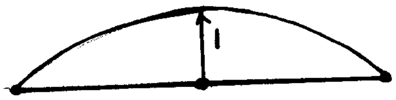
ACTUAL ELEMENT



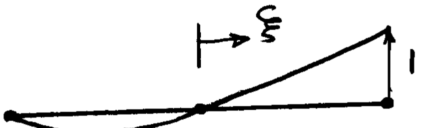
PARENT ELEMENT



$N_1 = \frac{1}{2}(-\xi + \xi^2)$



$N_2 = 1 - \xi^2$



$N_3 = \frac{1}{2}(\xi + \xi^2)$

We first form the shape functions for the Parent Element. These shape function can be formed by intuition or more rigorously by Lagrangian Interpolation .. eg. for shape function N_1 at Node 1 we have by Lagrange Interpolation.

$$N_1(\xi) = \frac{(0-\xi)(1-\xi)}{(0-(-1))(1-(-1))} = \frac{-\xi(1-\xi)}{1(2)} = \frac{1}{2}(-\xi+\xi^2)$$

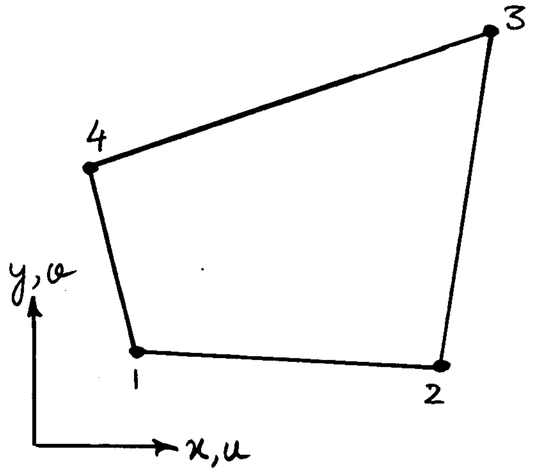
We can now express Global Coordinates in terms of Natural Coordinates as

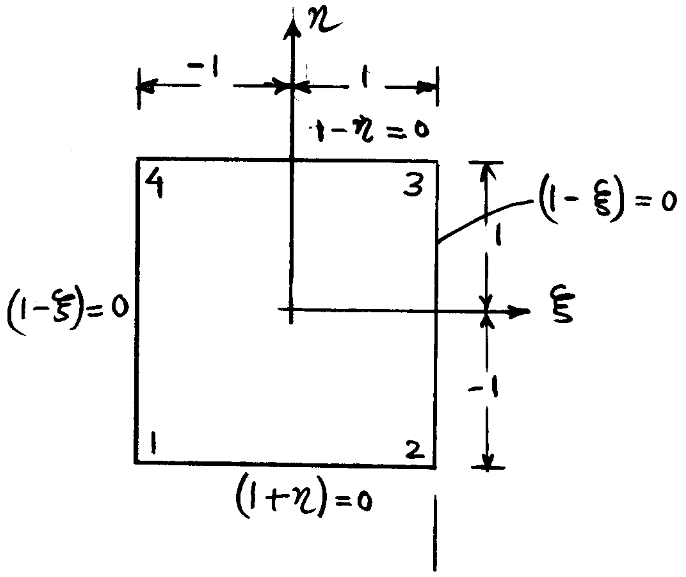
$$x = [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$\text{or} \quad x = \sum_{i=1}^{3} N_i x_i$$

Natural Coordinates & Interpolation in 2-Dimensions

Consider the quadrilateral element shown below which we want to map onto the parent element shown as well.





Shape Function for Node 1
can be developed by multiplying the equations for sides (2-3) and (3-4). This will ensure that the value of shape function is zero on lines 2-3 and 3-4 of the parent element.

$N_1' = (1-\xi)(1-\eta)$

N_1 @ $\xi=-1, \eta=-1 \quad = (1-(-1))(1-(-1))$

$= 2 \times 2 \quad = 4$

Therefore will need to Normalize N_1 by dividing by 4

$$\boxed{N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta)}$$

Nataaral Coordinates & Interpolation in 2-Dimensions

We can determine the remaining shape functions for the quadrilateral element in simillar manner. The complete set of shape functions is given below for the element

$$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta)$$
$$N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$
$$N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)$$
$$N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

Shape Functions for 4 Noded Quad Element

The Global Coordinates of any point within the Actual Element can now be expressed in terms of the Natural Coordinates of the Parent Element as follows:

$$x = \sum_{i=1}^{4} N_i \, x_i \qquad \text{or} \qquad \begin{Bmatrix} x \\ y \end{Bmatrix} = [N]\{c\}$$
$$y = \sum_{i=1}^{4} N_i \, Y_i$$

The Displacement Field within the Element Can be Expressed in terms of Nodal Displacements as:

$$u(x,y) = \sum_{i=1}^{4} N_i \, u_i \qquad \text{or} \qquad \begin{Bmatrix} u \\ v \end{Bmatrix} = [N]\{d\}$$
$$v(x,y) = \sum_{i=1}^{4} N_i \, v_i$$

Natural Coordinates and Interpolation in 2-Dimensions

Where,

$$\{c\} = [x_1 \quad y_1 \quad x_2 \quad y_2 \quad x_3 \quad y_3 \quad x_4 \quad y_4]^T$$

$$\{d\} = [u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4]^T$$

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

Comments on Requirements for Shape Functions

The shape functions constructed for an element should have the following attributes

1. They should represent a displacement field such that they can portray rigid body motion

2. The interpolation functions should be such that states of Constant stress can be portrayed by the element.

3. The displacement field represented by the shape functions should be such that "interelement compatibility" of displacements at common boundary between two elements is preserved. Elements which satisfy the compatibility requirement are called "Conforming Elements"

4. The displacement field represented by the shape functions should be such that it is "balanced" in terms of x and y (or ξ and η)

For example if we had to choose between including one term from the quadratic terms x^2, y^2, xy, the better choice is xy because of balance in x and y it represents.

Requirements 1 & 2 are necessary for convergence as mesh is refined as rigid body motions and constant stress states are achieved as the mesh is refined

For Rigid Body Motion we must have: $\sum_{i}^{n} N_i = 1$

Some Rules Governing relationship between Global Coordinates and Natural Coordinates

In isoparametric elements the element geometry and the displacement field is interpolated using the same shape functions. Thus for element geometry interpolation and displacement interpolation we have:

$$x = \sum_{i=1}^{q} h_i x_i \quad , \quad y = \sum_{i=1}^{q} h_i y_i \quad , \quad z = \sum_{i=1}^{q} h_i z_i$$

$$u = \sum_{i=1}^{q} h_i u_i \quad , \quad v = \sum_{i=1}^{q} h_i v_i \quad , \quad z = \sum_{i=1}^{q} h_i Z_i$$

where h_i are functions of natural coordinates ξ, η, ζ

For calculating strains we need to calculate

$\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$ etc

Now x, y, z can be expressed as

$$u = f_1(\xi, \eta, \zeta) \quad , \quad y = f_2(\xi, \eta, \zeta) \quad , \quad z = f_3(\xi, \eta, \zeta)$$

The inverse relation is

$$\xi = f_4(x, y, z) \quad , \quad \eta = f_5(x, y, z) \quad , \quad \zeta = f_6(x, y, z)$$

Using chain Rule we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial}{\partial \zeta}\frac{\partial \zeta}{\partial x}$$

Some Rules regarding relationship between Global & Natural Coordinates

To evaluate $\frac{\partial}{\partial x}$ we need to evaluate $\frac{\partial \xi}{\partial x}$ $\frac{\partial \eta}{\partial x}$ and $\frac{\partial \zeta}{\partial x}$. These inverse relations are difficult to form directly. However, using chain rule we can write

$$\left\{\begin{matrix} \frac{\partial}{\partial \xi} \\[2ex] \frac{\partial}{\partial \eta} \\[2ex] \frac{\partial}{\partial \zeta} \end{matrix}\right\} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\[2ex] \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\[2ex] \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix} \left\{\begin{matrix} \frac{\partial}{\partial x} \\[2ex] \frac{\partial}{\partial y} \\[2ex] \frac{\partial}{\partial z} \end{matrix}\right\} \qquad \text{———} \quad \textcircled{A}$$

or in Matrix Form

$$\left\{\frac{\partial}{\partial X}\right\} = \left[\,J\,\right] \left\{\frac{\partial}{\partial x}\right\}$$

where J = Jacobian Matrix relating the natural coordinate derivatives to the Global Coordinates Derivatives

Jacobian Matrix can be found by taking derivatives of coordinate interpolation relations

$$x = \sum h_i x_i \, , \quad y = \sum h_i y_i \, , \quad z = \sum h_i z_i$$

Some Rules Governing relationship between Global Coordinates and Natural Coordinates

To compute derivatives in global coordinates we invert Equation Ⓐ

$$\begin{Bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{Bmatrix} \quad \text{———} \quad Ⓑ$$

A differential area or volume in Global Coordinates is related to area and volume in natural coordinates by following relations

$$dx\,dy = |\mathrm{Det}\,J| \cdot d\xi\, d\eta \quad \text{———} \quad Ⓒ$$

$$dx\,dy\,dz = |\mathrm{Det}\,J|\, d\xi\, d\eta\, d\zeta$$