Some Rules Governing relationship between Global Coordinates and Natural Coordinates

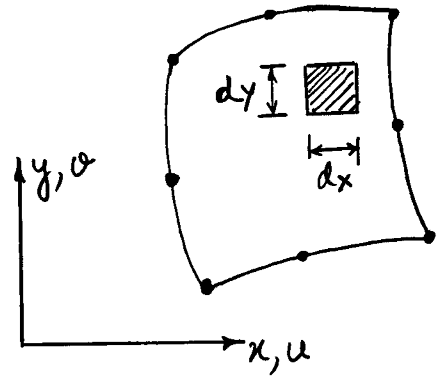
To compute derivatives in global coordinates we invert Equation (A)

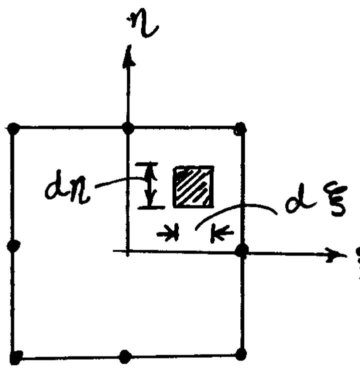
$$\begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \\ \dfrac{\partial}{\partial \zeta} \end{Bmatrix} \qquad \text{—— (B)}$$

A differential area or volume in Global coordinates is related to area and volume in natural coordinates by following relations

$$dx\,dy = |\text{Det } J| \cdot d\xi\, d\eta \qquad \text{—— (C)}$$

$$dx\,dy\,dz = |\text{Det } J|\, d\xi\, d\eta\, d\zeta$$





ISOPARAMETRIC ELEMENTS

We have seen that it is possible to express the displacements within an element in terms of element natural coordinates ξ, η, ζ i.e

$$u(x,y,z) = \sum N_i(\xi,\eta,\zeta)\, U_i$$

$u(x,y,z)$ = Displacements in Global Coordinates

$N_i(\xi,\eta,\zeta)$ = Element Shape Functions in Local Natural Coordinates

U_i = Element Nodal Displacements

Also we have seen that coordinates of any point within the actual element in Global Coordinates can be expressed in terms of the Local Natural Coordinates i.e The Shape Functions in Natural Coordinates can be used to describe the element geometry in global coordinates as follows:

$$x(x,y,z) = \sum N_i(\xi,\eta,\zeta)\, X_i$$

$$y(x,y,z) = \sum N_i(\xi,\eta,\zeta)\, Y_i$$

$$z(x,y,z) = \sum N_i(\xi,\eta,\zeta)\, Z_i$$

Thus it is possible to express element displacement field and geometry using shape functions in Natural Coordinates:

$$\boxed{\begin{array}{ll} [u, v, w]^T = [N(\xi,\eta,\zeta)]\,\{U_i\} & \text{— Displacement Field} \\ [x, y, z]^T = [\tilde{N}(\xi,\eta,\zeta)]\,\{X\} & \text{— Element Geometry} \end{array}}$$

ISOPARAMETRIC ELEMENTS

* In the previous Equations $N(\xi,\eta,\zeta)$ and $\tilde{N}(\xi,\eta,\zeta)$ are shape functions in natural coordinates
* and the order of polynomials in the two shape functions can be different.

An element is called <u>Parametric</u> if its displacement field and geometry is expressed in terms of parametric
* elements in which natural coordinates vary between range -1 to $+1$.

An element is called <u>Isoparametric</u> (having same parameter) if Shape functions used for describing the displacement field $[N]$ are identical to the shape functions $[\tilde{N}]$ used to describe the element geometry

$$\{u\} = [N]\{u_i\} \qquad \{x\} = [\tilde{N}]\{X\}$$

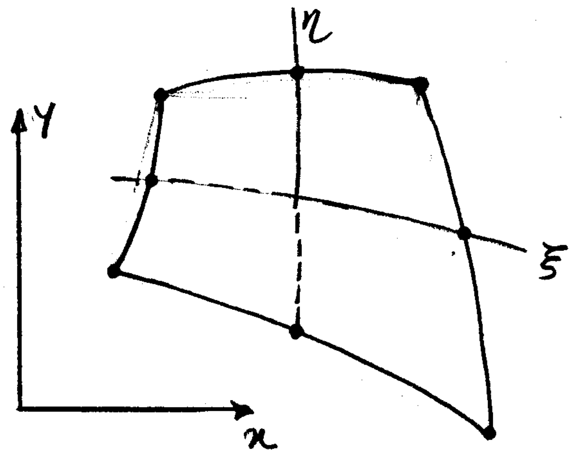
If $[N] = [\tilde{N}]$ Then Element is Isoparametric

If $[\tilde{N}]$ is of lower order than $[N]$ then the element is called <u>Subparametric</u>.

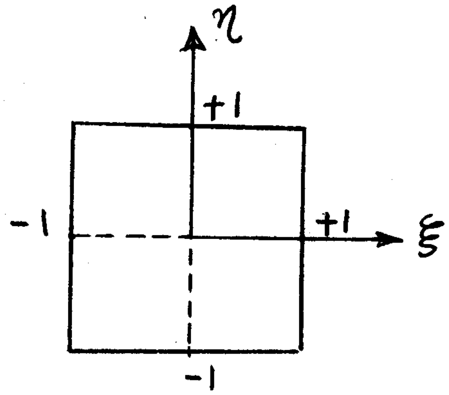
If $[\tilde{N}]$ is of a higher order than $[N]$ then the element is called <u>Superparametric</u>

8-Noded Quadratic Isoparametric Element

8-Noded Quadratic Isoparametric Element (Q8) obtained by adding a node at the midsides of the sides of the 4-Noded Quadratic Elements. The Q8 element can model curved geometries quite well. These types of elements are also called "Serendipity Elements"



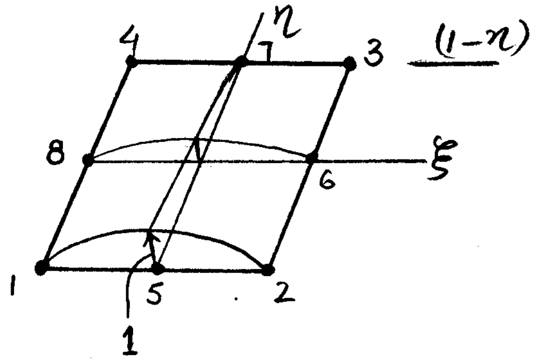
8-Noded Q8 Serendipity Element in Global Cartesian Coordinates



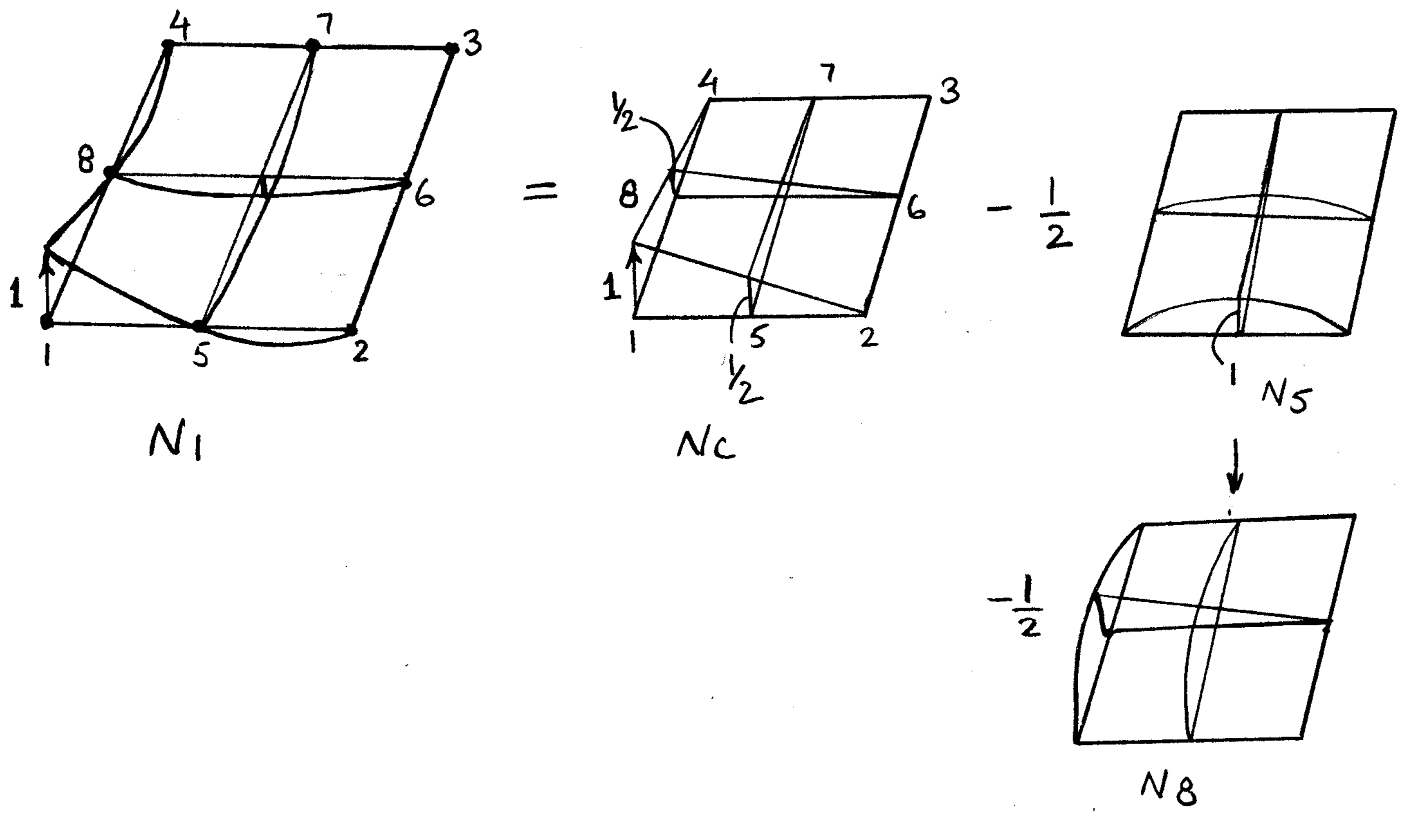
8-Noded Q8 Serendipity Element in Natural Coordinates

Q8 Serendipity Element

An intuitive way of deriving the shape functions of this element is given below.



$\phi_5 = (1-\xi^2)$



$N5$ = Eqn of line 1-2 X Eqn of line 3-4

$= (1-\xi^2)(1-\eta)$

N_5 @ Node 5 = $(1-0)(1-(-1)) = 2$ $\Rightarrow$ Normalize by dividing by 2

$N5 = \frac{1}{2}(1-\xi^2)(1-\eta)$

8-Noded Serendipity Element

Simillarly,

$$N6 = \frac{1}{2}(1+\eta^2)(1+\xi) \quad = \frac{1}{2}(1+\xi)(1-\eta^2)$$

$$N7 = \quad\quad = \frac{1}{2}(1-\xi^2)(1+\eta)$$

$$N8 = \quad\quad = \frac{1}{2}(1-\xi)(1-\eta^2)$$

N_1 can be constructed by intuition

we see that

$$N_1 = \quad N_c \;-\tfrac{1}{2}\, N5 - \tfrac{1}{2}\, N8$$

$$N_c = \frac{1}{4}(1-\xi)(1-\eta)$$

$$\Rightarrow \quad N_1 = \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{2}(N5 + N8)$$

$$= \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{2}\left\{(1-\xi^2)(1-\eta) + (1-\xi)(1-\eta^2)\right\}$$

Simillarly N_2, N_3 & N_4 Can be Constructed

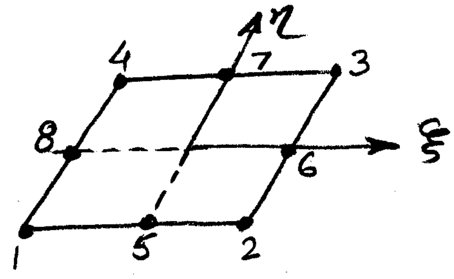
$$N2 = \frac{1}{4}(1+\xi)(1-\eta) - \frac{1}{2}(N5 + N6)$$

$$N3 = \frac{1}{4}(1+\xi)(1+\eta) - \frac{1}{2}(N6 + N7)$$

$$N4 = \frac{1}{4}(1-\xi)(1+\eta) - \frac{1}{2}(N7 + N8)$$

8 - Noded Serendipity Element

The Shape Functions of the 8-Node Serendipity element are summarized below:



$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{2}(N_5 + N_8)$$

$$N_2 = \frac{1}{4}(1+\xi)(1-\eta) - \frac{1}{2}(N_5 + N_6)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta) - \frac{1}{2}(N_6 + N_7)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta) - \frac{1}{2}(N_7 + N_8)$$

$$N_5 = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$N_6 = \frac{1}{2}(1+\xi)(1-\eta^2)$$

$$N_7 = \frac{1}{2}(1-\xi^2)(1+\eta)$$

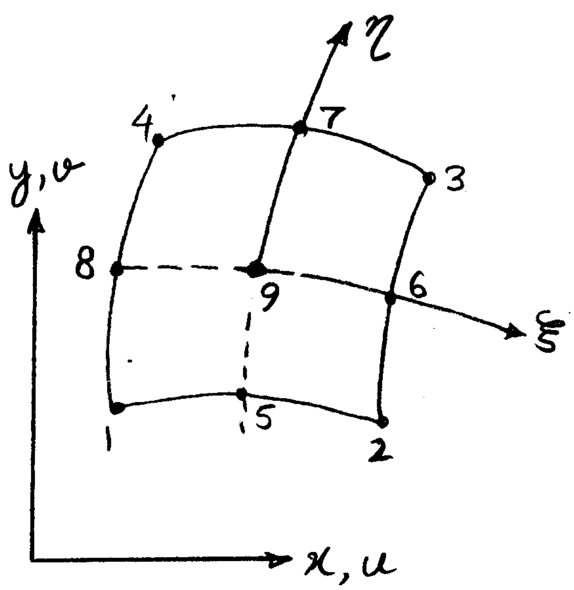
$$N_8 = \frac{1}{2}(1-\xi)(1-\eta^2)$$

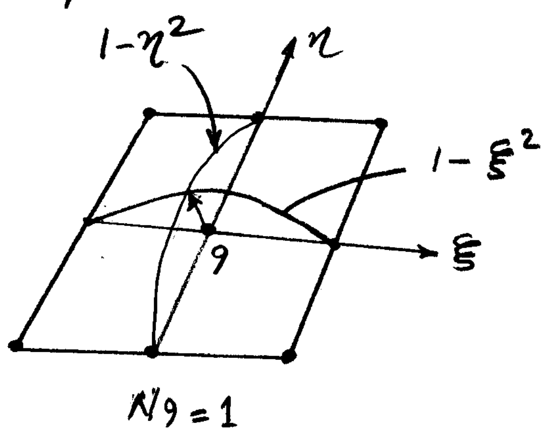
9- Noded (Q9) Quadratic Lagrangian Element

Addition of an internal node in the center of the 8-node (Q8) Serendipity element results in formation of 9-noded Lagrangian Element.

The element geometry is completely defined by the 8 nodes on the element boundary. The 9th internal node is not required for element geometry definition.

Note that although the 9th node is used in describing the displacement field and not the element geometry, the element nevertheless is isoparametric because the polynomial used for geometry description and displacement field is quadratic.





The Q9 element is called a Lagrangian element because the shape functions for the element can be developed by Lagrangian Interpolation. However, the shape functions can be easily developed using the intuitive approach.

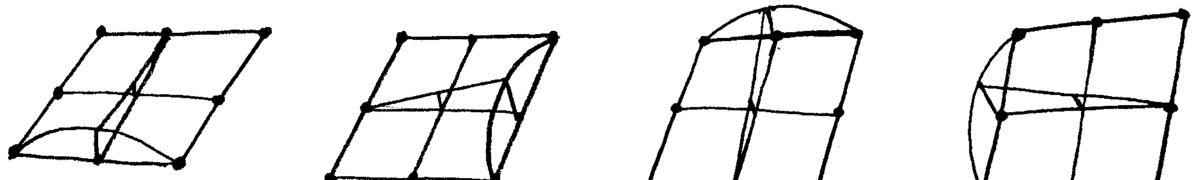
9-Noded (Q9) Lagrangian Element

The shape function associated with node 9 can be obtained by taking a product of two one dimensional Lagrangian interpolation functions

$$N_9(\xi,\eta) = (1-\xi^2)(1-\eta^2) \quad \text{— Bubble function.}$$

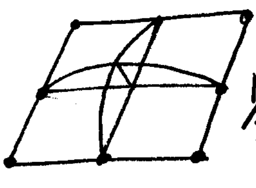
The first 8 shape functions N_1 to N_8 can be obtained by modifying the shape functions of the 8-Noded Serendipity element such that these shape functions have a value equal to zero at the location of node 9 ($\xi = \eta = 0$)

N_5 through N_8 have a value equal to $+\frac{1}{2}$ at N_9 location. $N_5 = \frac{1}{2}(1-\xi^2)(1-\eta) = \frac{1}{2}$ @ $\xi = \eta = 0$



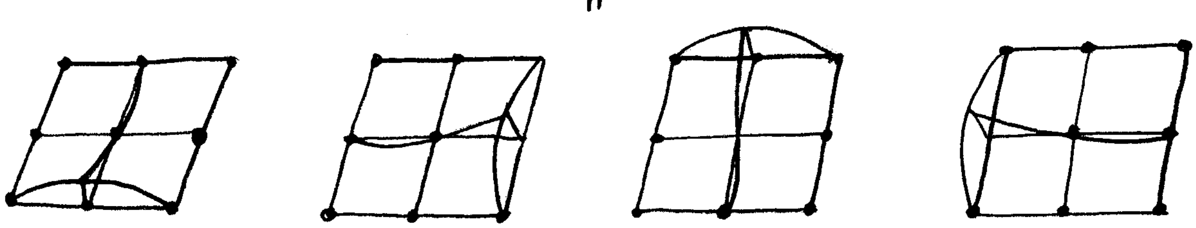
N5-Q8 N6-Q8 N7-Q8 N8-Q8

|



$\frac{1}{2}(N_9 - Q_9)$

||

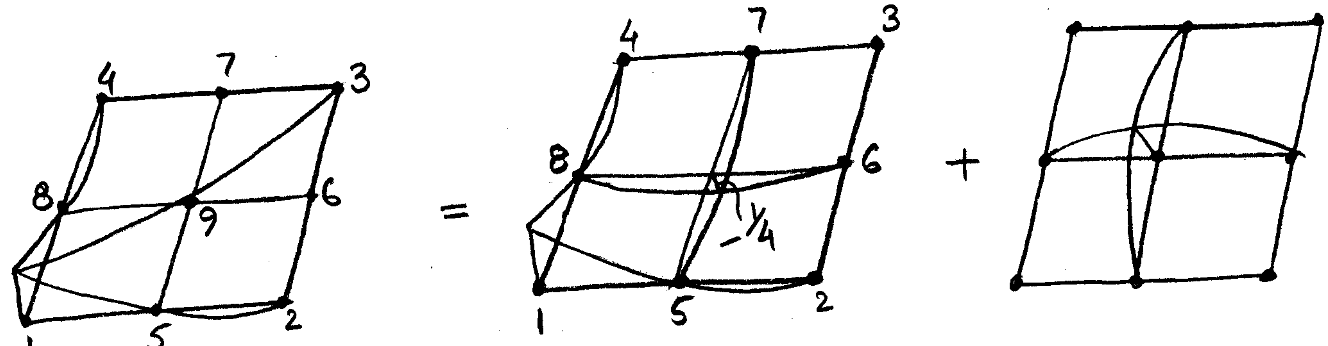


N5-Q9 N6-Q9 N7-Q9 N8-Q9

9-Noded (Q9) Lagrangian Element

Thus if we subtract $-\frac{1}{2}$ times N9 shape function from $N5 \rightarrow N8$ of Q8 Serendipity element we will obtain $N5 - N8$ of Q9 Lagrangian Element.

Simillarly, $N1 \rightarrow N4$ of Q8 Serendipity Element have a value of $-\frac{1}{4}$ at Q9 $(\xi = \eta = 0)$. Therefore, $N1 \rightarrow N4$ of Q9 can be obtained by adding $\frac{1}{4}$ Q9 to the $N1 \rightarrow N4$ of Q8



$N_1 - Q_9$ $\qquad$ $N_1 - Q_8$ $\qquad$ $\frac{1}{4} N_9 - Q_9$

$$\overset{Q9}{N_1 \rightarrow N_4} = \overset{Q8}{N_1 \rightarrow N_4} + \frac{1}{4} \overset{Q9}{N_9}$$

Thus,

$$N_{5,Q_9} = N_{5,Q_8} + \frac{1}{4} N_{9,Q_8}$$

$$N_{5,Q_9} = \frac{1}{2}(1-\xi^2)(1-\eta) - \frac{1}{2}(1-\xi^2)(1-\eta^2)$$

$$N_{1,Q_9} = N_{1,Q_8} + \frac{1}{4} N_9$$

$$= \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{2}\left[\frac{1}{2}(1-\xi)(1-\eta^2) + \frac{1}{2}(1-\xi^2)(1-\eta)\right]$$

$$+ \frac{1}{4} N_9$$

9-Noded Q9 Lagrangian Element

$$N_{1,Q9} = \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{2}\left[\frac{1}{2}(1-\xi)(1-\eta^2) - \frac{1}{2}N_9\right]$$
$$- \frac{1}{2}\left[\frac{1}{2}(1-\xi^2)(1-\eta) - \frac{1}{2}N_9\right]$$
$$- \frac{1}{4}N_9$$

Complete set of Shape functions for Q9 Lagrangian Element are given below:

Shape Function $Q8 \rightarrow Q9$	Include only if Node i is present in the Element				
	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$	$-\frac{1}{2}N_5$			$-\frac{1}{2}N_8$	$-\frac{1}{4}N_9$
$N_2 = \frac{1}{4}(1+\xi)(1-\eta)$	$-\frac{1}{2}N_5$	$-\frac{1}{2}N_6$			$-\frac{1}{4}N_9$
$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$		$-\frac{1}{2}N_6$	$-\frac{1}{2}N_7$		$-\frac{1}{4}N_9$
$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$			$-\frac{1}{2}N_7$	$-\frac{1}{2}N_8$	$-\frac{1}{4}N_9$
$N_5 = \frac{1}{2}(1-\xi^2)(1-\eta)$					$-\frac{1}{2}N_9$
$N_6 = \frac{1}{2}(1+\xi)(1-\eta^2)$					$-\frac{1}{2}N_9$
$N_7 = \frac{1}{2}(1-\xi^2)(1+\eta)$					$-\frac{1}{2}N_9$
$N_8 = \frac{1}{2}(1-\xi)(1-\eta^2)$					$-\frac{1}{2}N_9$
$N_9 = (1-\xi^2)(1-\eta^2)$					$-\frac{1}{2}N_9$

Numerical Integration

In Isoparametric elements and other higher order element it is very difficult to analytically compute the stiffness matrices and equivalent nodal load vectors. Numerical integration offers a way by which these quantities can be calculated relatively easily

For element Stiffness Matrix we need to evaluate the integral

$$\int B^T D B \, d\xi \, d\eta \, d\zeta$$

This integral is evaluated numerically as follows:

$$\int B^T(\xi,\eta,\zeta) \; D \; B(\xi,\eta,\zeta) \, |\mathrm{Det}\, J| \, d\xi \, d\eta \, d\zeta$$

$$= \int F(\xi,\eta,\zeta) \, d\xi \, d\eta \, d\zeta \quad = \sum_i \alpha_{ijk} F(\xi_i, \eta_j, \zeta_k) + R_n$$

where α_{ijk} are weighting Constants and $F(\xi_i, \eta_i, \zeta_k)$ is the $B^T D B$ matrix evaluated at a specified location

ξ_i, η_i, ζ_k

Simillarly other integrals that need evaluation are integrals which equivalent Nodal Force Vectors.

$$\int N^T(\xi,\eta,\zeta) \, f_B \, |\mathrm{Det}\, J| \, d\xi \, d\eta \, d\zeta \quad \text{— Body Force Vector}$$

and

$$\int N^T(\xi,\eta,\zeta) \, f_S \, |\mathrm{Det}\, J| \, d\xi \, d\eta \quad \text{— Surface Traction Vector.}$$

Numerical Integration

In general integrals in one, two and three dimensions can be evaluated numerically as follows

$$\int F(\xi)\, d\xi = \sum \alpha_i F(\xi_i) + R_n$$

$$\iint F(\xi,\eta)\, d\xi\, d\eta = \sum_{ij} \alpha_{ij} F(\xi_i,\eta_j) + R_n$$

$$\iiint F(\xi,\eta,\zeta)\, d\xi\, d\eta\, d\zeta = \sum_{ijk} \alpha_{ijk} F(\xi_i \eta_j \zeta_k) + R_n$$

where

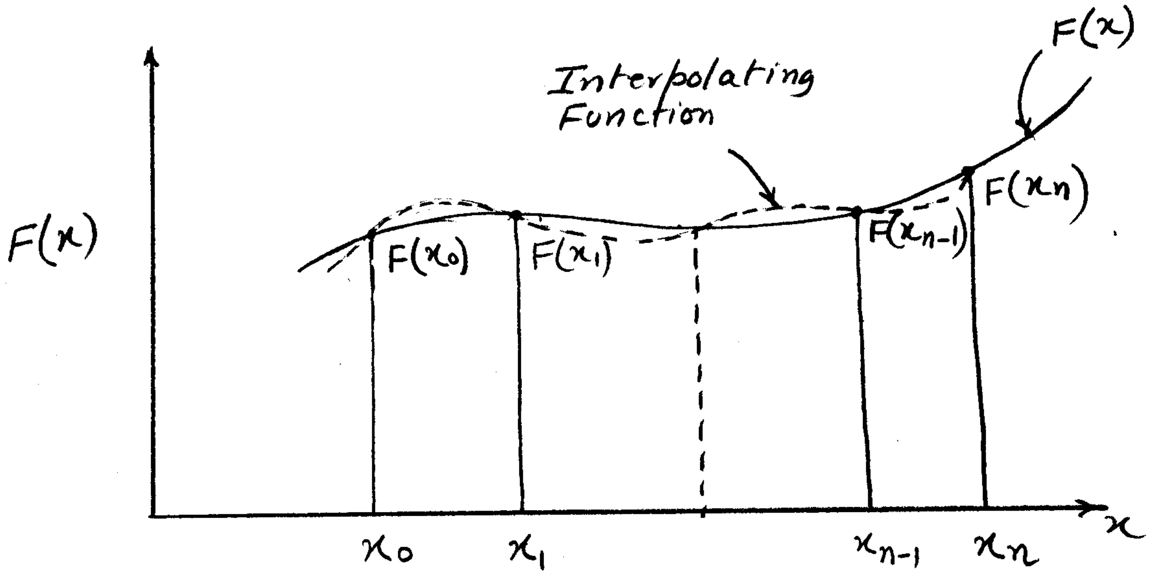
$\alpha_i, \alpha_{ij}, \alpha_{ijk}$ = weighting Constants

$F(\xi_i), F(\xi_i,\eta_j)$, $F(\xi_i,\eta_j,\zeta_k)$ = Numerical Value of the Function to integrated evaluated at Specified locations.

R_n = Residual or error

In the above equations the Error or Residual is often neglected

Newton-Cotes Formulas for One-Dimensional Integration



The figure above shows a function over the range $x_0 \rightarrow x_n$. If the value of this function is known at n discrete locations then an interpolating function can be fitted through the known data pts to approximate the function through Lagrangian interpolation functions or through use of other shape functions.

$$F(x) \simeq \phi(x) = \sum_{i=0}^{n} l_i(x)\, F(x_i)$$

where $\phi(x)$ = interpolating function

$l_i(x)$ = Interpolation Shape Functions

$F(x_i)$ = Values of Function at x_i locations

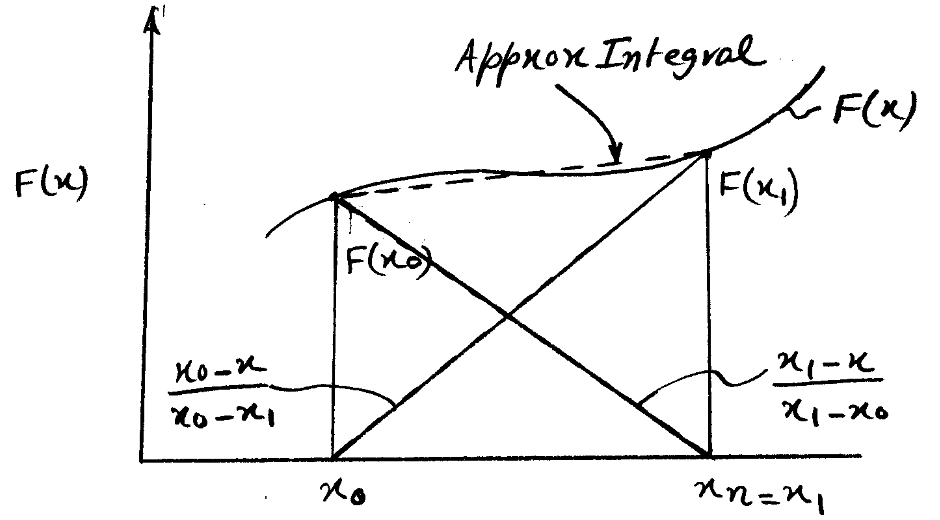
Then

$$\int_{x_0}^{x_n} F(x)\,dx \simeq \int_{x_0}^{x_n} \phi(x)\,dx = \sum_{i=0}^{n}\left[\int_{x_0}^{x_n} l_i(x)\,dx\right] F(x_i) + \underbrace{R_n}_{\text{Neglect}}$$

$$= (x_n - x_0)\sum_{i=0}^{n} C_i^n F_i$$

C_i^n = Newton-Cotes Constants for Integration.

Numerical Integration



Example :

Newton - Cotes integration rule for 2 pts. or 1 interval

$$F(x) = \sum_{i=0}^{1} l_i(x)\, F(x_i)$$

$$l_1(x) = \frac{x_1 - x}{x_1 - x_0} \quad , \quad l_2(x) = \frac{x_0 - x}{x_0 - x_1}$$

$$\int_{x_0}^{x_1} F(x)\, dx = \sum \left[\int_{x_0}^{x_1} l_i(x)\, dx \right] F(x_i)$$

$$= \left[\int_{x_0}^{x_i} \frac{x_1 - x}{x_1 - x_0}\, dx \right] F(x_0) + \left[\int_{x_0}^{x_1} \frac{x_0 - x}{x_0 - x_1}\, dx \right] F(X_1)$$

$$= \frac{1}{x_1 - x_0} \left| x_1 x - \frac{x^2}{2} \right|_{x_0}^{x_1} F(x_0) + \frac{1}{x_0 - x_1} \left| x_0 x - \frac{x^2}{2} \right|_{x_0}^{x_1} F(x_1)$$

$$= \frac{1}{x_1 - x_0} \left[x_1^2 - \frac{x_1^2}{2} - x_1 x_0 + \frac{x_0^2}{2} \right] F(x_0)$$

$$+ \frac{1}{x_0 - x_1} \left[x_0 x_1 - \frac{x_1^2}{2} - x_0^2 + \frac{x_0^2}{2} \right] F(x_1)$$

$$= \frac{1}{(x_1 - x_0)} \frac{(x_1 - x_0)^2}{2} F(x_0) - \frac{1}{(x_0 - x_1)} \frac{(x_0 - x_1)^2}{2} F(x_1)$$

Nomerical Integration

Example (Contd.) Newton-Cotes rule for 2 pt integration or 1 interval

$$\int_{x_0}^{x_1} F(x)\,dx = \frac{x_1 - x_0}{2} \cdot F(x_0) - \frac{(x_0 - x_1)}{2} F(x_1)$$

$$= \frac{(x_1 - x_0)}{2} F(x_0) + \frac{(x_1 - x_0)}{2} F(x_1)$$

$$\boxed{\int_{x_0}^{x_1} F(x)\,dx = (x_1 - x_0)\left[\frac{1}{2} F(x_0) + \frac{1}{2} F(x_1)\right]}$$

↳ Newton-Cotes Rule for 2 Pt Integration or 1 Interval

Also Known as "Trapezoidal Rule".

The Newton-Cotes Coefficients for various Intervals of Integration are summarized below:

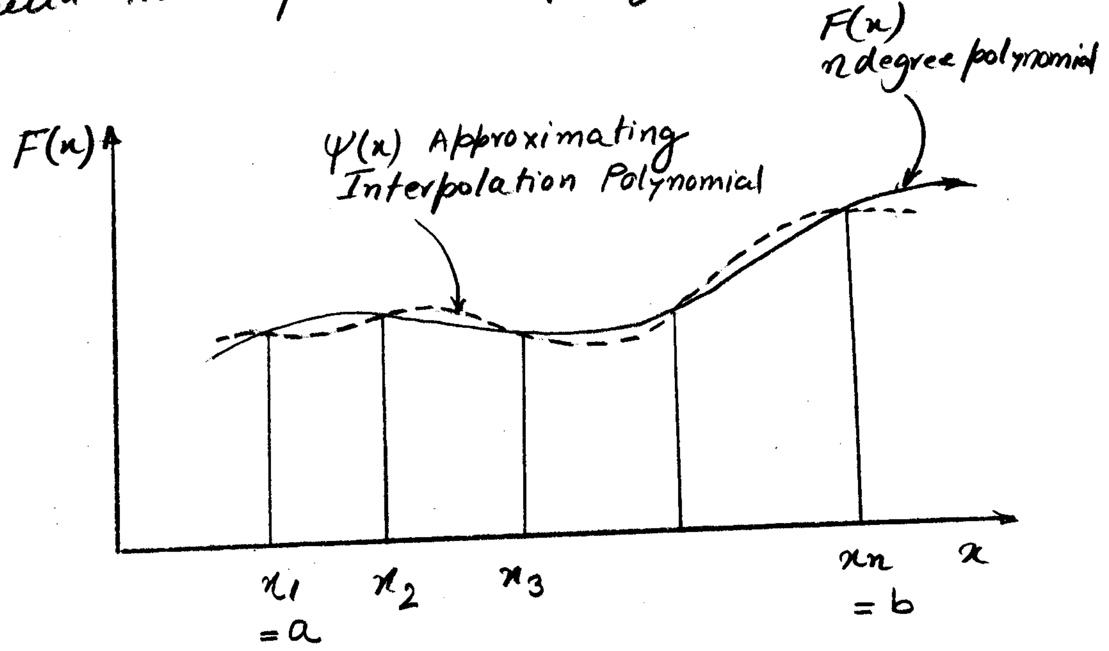
NEWTON-COTES COEFFICIENTS FOR INTEGRATION

No. of Intervals	No of Pts.	C_0	C_1	C_2	C_3	C_4	C_5	Error Upper Bound
1	2	$\frac{1}{2}$	$\frac{1}{2}$					$10^{-1}(x_1 - x_0)^3 F''(x)$
2	3	$\frac{1}{6}$	$\frac{4}{6}$	$\frac{1}{6}$				$10^{-3}(x_1 - x_0)^5 F^{iv}(x)$
3	4	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$			$10^{-3}(x_1 - x_0)^5 F^{iv}(x)$
4	5	$\frac{7}{90}$	$\frac{32}{90}$	$\frac{12}{90}$	$\frac{32}{90}$	$\frac{7}{90}$		$10^{-6}(x_1 - x_0)^7 F^{iv}(x)$
5	6	$\frac{19}{288}$	$\frac{75}{288}$	$\frac{50}{288}$	$\frac{50}{288}$	$\frac{75}{288}$	$\frac{19}{288}$	$10^{-6}(x_1 - x_0)^7 F^{iv}(x)$

Numerical Integration

Gauss Quadrature

Newton-Cotes Integration attempts to improve integration accuracy by keeping the sampling pts locations fixed and adjusting the weight associated with sampling location. Gauss Integration/Quadrature attempts to improve integration accuracy by finding optimum function sampling locations as well as finding optimum weighting constants associated with the improved sampling locations.



The basic assumption in Gauss Quadrature is that an Integral of a Polynomial or a Function can be expressed as:

$$\int_a^b F(x)\,dx = \alpha_1 F(x_1) + \alpha_2 F(x_2) + \cdots \alpha_n F(x_n) + R_n$$

where both the weights $\alpha_1, \cdots \alpha_n$ and the sampling pts locations $x_1, x_2 \cdots x_n$ are unknown variables that need to be found subject to requirement that the error is minimum. Thus there are $2n$ variables to be determined compared to n variables in Newton-Cotes Integration.

Gauss Quadrature

Just as in Newton-Cotes Integration, we use a Lagrangian Interpolation polynomial to approximate the Function $F(x)$

$$F(x) \simeq \psi(x) = \sum_{j=1}^{n} l_j(x)\, F_j(x_j)$$

where the n sampling locations are unknown. For determining the position of sampling locations $x_1, x_2 --- x_n$ we define a function $P(x)$

$$P(x) = (x-x_1)(x-x_2)----(x-x_n)$$

which is a Polynomial of order "n". The polynomial $P(x) = 0$ at the sampling locations $x_1, x_2 --- x_n$

Therefore we can write

$$F(x) = \psi(x) + P(x)\left(\beta_0 + \beta_1 x + \beta_2 x^2 + ---\right)$$

$$F(x) = \sum_{j=1}^{n} l_j(x)\, F_j(x_j) + P(x)\left(\beta_0 + \beta_1 x + \beta_2 x^2 + ---\right)$$

Integrating $F(x)$ we have

$$\int_a^b F(x)\,dx = \sum_{j=1}^{n} F_j\left[\int_a^b l_j(x)\,dx\right] + \sum_{j=0}^{\infty} \beta_j\left[\int_a^b x^j\, P(x)\,dx\right] \quad \text{— (A)}$$

The unknowns $x_1, x_2 --- x_n$ can now be determined from the condition

$$\int_a^b P(x)\, x^k\,dx = 0 \quad , \quad K = 0, 1, 2 --- n-1 \quad \text{— (B)}$$

Gauss Quadrature

It is noted that in Equation (A) the first term contains functions of order $(n-1)$ and lower. Whereas the second term contains functions of order n and higher.

In Equation (A) a polynomial of order n has been approximated by integrating a polynomial of order

$x^{n-1} P(x^n) \Rightarrow$ order $n+n-1 = 2n-1$

The sampling pts and integration wts depend upon the interval $a \to b$. To make the calculations general relate the interval $a \to b$ to natural coordinates interval $-1 \to +1$ and deduce the sampling pts and weights for any interval. If x_i is a sampling pt and α_i is the weight associated with it in the interval $-1 \to +1$, the corresponding sampling pt and weight in integration from interval $a \to b$ are

$$\underbrace{\frac{a+b}{2} + \frac{b-a}{2} x_i}_{\text{location}} \qquad \text{and} \qquad \underbrace{\frac{b-a}{2} \alpha_i}_{\text{weight}}$$

The locations of sampling pts are obtained from equation (B) $\int_a^b P(x) x^k dx = 0 \equiv \int_{-1}^{+1} P(\xi) \xi^k d\xi = 0$

The Integral can then be evaluated by

$$\int_a^b F(x)\, dx = \int_{-1}^{+1} F(\xi)\, d\xi = \alpha_1 F(\xi_1) + \alpha_2 F(\xi_2) + \cdots \alpha_n F(\xi_n)$$

Gauss Quadrature

The sampling locations and the corresponding weight constants for various orders of Gauss Quadrature are given in the table below:

SAMPLING LOCATIONS AND WEIGHTS FOR GAUSS QUADRATURE

No. of Sampling Pts	Sampling Location (ξ)	Weights
One Pt. Formula	0.000,000,000,0	2.000,000,000,0
2-Point Formula	±0.577,350,269,2	1.000,000,000,0
3-Point Formula	0.000,000,000,0 ±0.774,596,669,2	0.888,888,888,9 0.555,555,555,5
4-Point Formula	±0.339,981,043,5 ±0.861,136,311,6	0.652,145,154,8 0.347,854,845,1

The weights associated with sampling pt locations are found by substituting the interpolating polynomial $\psi(\xi) = \sum_{j=1}^{n} F_j l_j(\xi)$ in $\int_a^b F(\xi)\,d\xi = \sum_{j=1}^{n} \int_a^b F_j l_j(\xi)\,d\xi$ and carrying out the integration.

Since the sampling pts are now known the polynomial $\psi(\xi)$ is known. Therefore:

$$\boxed{\alpha_j = \int_{-1}^{+1} l_j(\xi)\,d\xi \qquad j = 1, 2, \dots n}$$

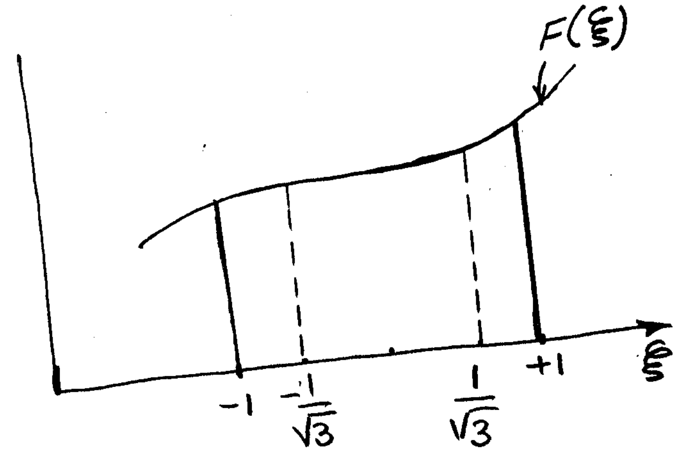
Example Problem
on Derivation of Sampling Pt
Location an Weights

Problem: Derive the sampling points and weights for two point Gauss Quadrature

The sampling pt locations are given by Equation

$$\int_a^b P(x)\, x^k\, dx = 0 \quad , \quad \text{where } K = 0, 1, 2 \cdots n-1$$

and $P(x)$ = Interpolating polynomial $= 0$ at Sampling Locations



$$P(\xi) = (\xi - \xi_1)(\xi - \xi_2)$$

For. Sampling pt location 1

$$\int_{-1}^{+1} P(\xi)\, \xi^k\, d\xi = \int_{-1}^{+1} (\xi - \xi_1)(\xi - \xi_2)\, \xi^0\, d\xi = 0$$

$$= \int_{-1}^{+1} (\xi - \xi_1)(\xi - \xi_2)\, d\xi = 0$$

$$= \int_{-1}^{+1} \left(\xi^2 - \xi\xi_2 - \xi\xi_1 + \xi_1\xi_2\right) d\xi = 0$$

$$= \int_{-1}^{+1} \left(\xi^2 - \xi(\xi_1 + \xi_2) + \xi_1\xi_2\right) d\xi = 0$$

Example Problem
Derivation of Gauss Pts and Weights

$$\left. \frac{\xi^3}{3} - \frac{\xi^2}{2}\cancelto{0}{(\xi_1 + \xi_2)} + \xi\,\xi_1\xi_2 \right|_{-1}^{+1} = 0$$

$$\Rightarrow \quad \frac{2}{3} + 2\xi_1\xi_2 = 0 \quad \Rightarrow \quad \boxed{\xi_1\xi_2 = -\frac{1}{3}}$$

For Sampling Pt No. 2

$$\int_{-1}^{+1} (\xi - \xi_1)(\xi - \xi)\,\xi\, d\xi = 0$$

$$\int_{-1}^{+1} \left(\xi^2 - \xi(\xi_1 + \xi_2) + \xi_1\xi_2\right)\xi\, d\xi = 0$$

$$\int_{-1}^{+1} \left(\xi^3 - \xi^2(\xi_1 + \xi_2) + \xi\,\xi_1\xi_2\right) d\xi = 0$$

$$\left. \cancelto{0}{\frac{\xi^4}{4}} - \frac{\xi^3}{3}(\xi_1 + \xi_2) + \cancelto{0}{\frac{\xi^2}{2}\,\xi_1\xi_2} \right|_{-1}^{+1} = 0$$

$$\left(-\frac{1}{3} - \frac{(-1)^3}{3}\right)(\xi_1 + \xi_2) = 0$$

$$\Rightarrow \quad \boxed{(\xi_1 + \xi_2) = 0}$$

Example Problem

Derivation of Gauss Pts and Weights

$$\left. \frac{\xi^3}{3} - \cancelto{0}{\frac{\xi^2}{2}(\xi_1+\xi_2)} + \xi\xi_1\xi_2 \right|_{-1}^{+1} = 0$$

$$\Rightarrow \frac{2}{3} + 2\xi_1\xi_2 = 0 \quad \Rightarrow \quad \boxed{\xi_1\xi_2 = -\frac{1}{3}}$$

For Sampling Pt No. 2

$$\int_{-1}^{+1} (\xi-\xi_1)(\xi-\xi)\,\xi\, d\xi = 0$$

$$\int_{-1}^{+1} \left(\xi^2 - \xi(\xi_1+\xi_2) + \xi_1\xi_2\right)\xi\, d\xi = 0$$

$$\int_{-1}^{+1} \left(\xi^3 - \xi^2(\xi_1+\xi_2) + \xi\xi_1\xi_2\right) d\xi = 0$$

$$\left. \cancelto{0}{\frac{\xi^4}{4}} - \frac{\xi^3}{3}(\xi_1+\xi_2) + \cancelto{0}{\frac{\xi^2}{2}\xi_1\xi_2} \right|_{-1}^{+1} = 0$$

$$\left(-\frac{1}{3} - \frac{(-1)^3}{3}\right)(\xi_1+\xi_2) = 0$$

$$\Rightarrow \boxed{(\xi_1+\xi_2) = 0}$$

Example Problem
Gauss Pts for 2-Point Gauss Formula

The two simultaneous equations for Gauss Pts location are as follows:

$$\xi_1 \xi_2 = -\frac{1}{3}$$

$$\xi_1 + \xi_2 = 0$$

$$\Rightarrow \quad \xi_1 = -\xi_2$$

$$-\xi_2^2 = -\frac{1}{3} \quad \Rightarrow \quad \boxed{\xi_2 = \frac{1}{\sqrt{3}} = 0.57735027}$$

$$\xi_1 = -\xi_2 \quad \Rightarrow \quad \boxed{\xi_1 = -\frac{1}{\sqrt{3}} = -0.57735027}$$

For weights associated with these sampling locations these are given by following formula

$$\boxed{\alpha_j = \int_{-1}^{+1} l_j(\xi)\, d\xi}$$

$$\alpha_1 = \int_{-1}^{+1} \frac{\xi - \xi_2}{\xi_1 - \xi_2}\, d\xi = \frac{1}{(\xi_1 - \xi_2)} \left| \frac{\xi^2}{2} - \xi\xi_2 \right|_{-1}^{+1}$$

$$= \frac{1}{\xi_1 - \xi_2} \left[-\xi_2 - \xi_2 \right]$$

$$= \frac{-2\xi_2}{\xi_1 - \xi_2} = \frac{-2\, {}^1\!/\!\sqrt{3}}{-{}^1\!/\!\sqrt{3} - {}^1\!/\!\sqrt{3}} = \frac{-{}^2\!/\!\sqrt{3}}{-{}^2\!/\!\sqrt{3}}$$

$$\boxed{\alpha_1 = 1.0}$$

Example Problem
Derivation of 2 Pt Gauss Rule

Similarly since $\xi_2 = -\xi_1$

we have for α_2 (weight associated with ξ_2)

$$\alpha_2 = \frac{2\xi_1}{\xi_1 + \xi_1} = 1.0$$

$$\boxed{\alpha_2 = 1.0}$$

Thus for 2-Pt Gauss Rule we have following Sampling Pt locations and weights

$$\boxed{\begin{array}{ll} \xi_1 = \dfrac{1}{\sqrt{3}}, & \alpha_1 = 1.0 \\ \xi_2 = -\dfrac{1}{\sqrt{3}}, & \alpha_2 = 1.0 \end{array}}$$