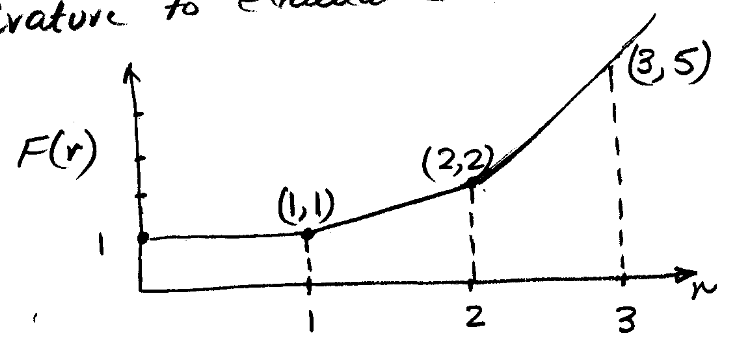
Comments on Numerical Integration

- Using Newton-Cotes Formulas we use $(n+1)$ equally spaced sampling points to exactly integrate a polynomial of at most order n
- using Gauss Quadrature we require n equally spaced sampling pts to exactly integrate a polynomial of at most order $(2n-1)$
- To integrate a quadratic polynomial ie $n=2$ will need $n+1=3$ sampling pts using Newton-Cotes
- Correspondingly in Gauss Quadrature polynomial order $2n-1=2 \Rightarrow n=\frac{3}{2}$ 2 pts.
  polynomial order $2n-1=3 \Rightarrow n=2$ pts.
- if polynomial order $2n-1=3$ Correspondingly Newton-Cotes will require 4 pts to exactly integrate a cubic polynomial

Example

Use 2pt Gauss Quadrature to evaluate

$$\int_0^3 (2^r - r)\,dr$$



$$\int_0^3 (2^r - r)\,dr = \alpha_1 F(r_1) + \alpha_2 F(r_2)$$

α_1, α_2 = weights

r_1, r_2 = sampling pts

$\alpha_1 = \alpha_2 = \frac{3-0}{2} \times 1 = \frac{3}{2}$, $r_1 = \frac{3}{2}\left(1 - \frac{1}{\sqrt{3}}\right) = 0.63397$

$r_2 = \frac{3}{2}\left(1 + \frac{1}{\sqrt{3}}\right) = 2.36603$

$$\int_0^3 (2^r - r)\,dr \simeq \frac{3}{2}\left(2^{0.63397} - 0.63397\right) + \frac{3}{2}\left(2^{2.36603} - 2.36603\right) = \boxed{5.56055}$$

Comments on Numerical Integration

Example Problem

Exact Solution

$$\int_0^3 (2^x - x)\,dx = \left| \frac{2^x}{\ln 2} - \frac{x^2}{2} \right|_0^3$$

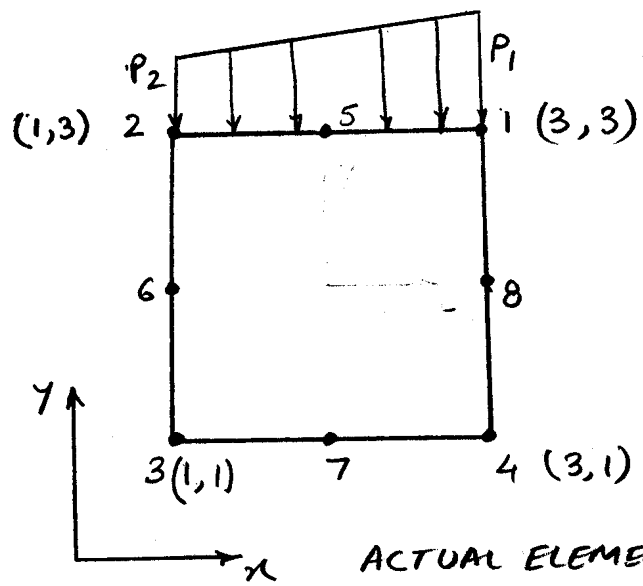
$$= \frac{2^3}{\ln 2} - \frac{2^0}{\ln 2} - \frac{3^2}{2}$$

$$= 11.54156 - 1.44269 - \frac{9}{2}$$

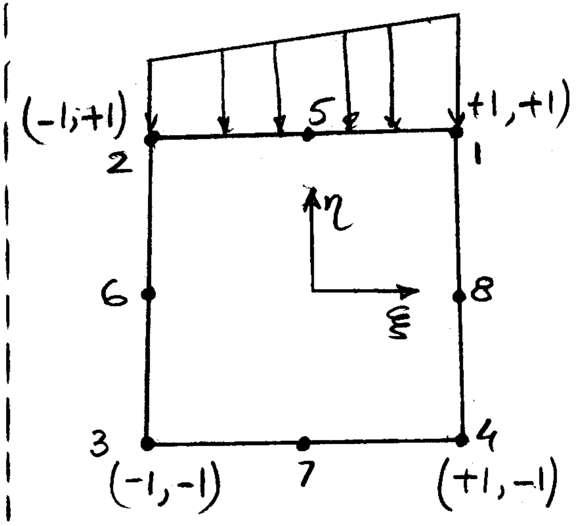
$$= \underset{\text{exact}}{5.59887} \simeq \underset{\text{Numerical}}{5.56055}$$

% error = 0.7%

Example Problem



ACTUAL ELEMENT



PARENT ELEMENT

For the 2-D element shown above calculate the Jacobian Matrix of Transformation from cartesian coordinates to natural coordinates.

Evaluate the consistent Nodal loads corresponding to the surface loading shown

$$\xi = \frac{x - \frac{x_1 + x_2}{2}}{(x_1 - x_2)/2} = \frac{2x - x_1 - x_2}{(x_1 - x_2)}$$

$$\eta = \frac{y - \frac{y_1 + y_4}{2}}{(y_1 - y_4)/2} = \frac{2y - y_1 - y_4}{(y_1 - y_4)}$$

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}$$

Example Problem

$$\begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\[2ex] \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\[2ex] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix}}_{[J] \text{ Jacobian Matrix}} \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\[2ex] \dfrac{\partial f}{\partial y} \end{Bmatrix}$$

$$\xi = \frac{2x - x_1 + x_2}{(x_1 - x_2)} = \frac{2x - 3 - 1}{(3-1)} = \frac{2x - 4}{2}$$

$$\Rightarrow \quad 2x - 4 = 2\xi$$

$$\Rightarrow \quad \boxed{x = \frac{2\xi + 4}{2} = \xi + 2}$$

$$\eta = \frac{2y - y_1 - y_4}{(y_1 - y_4)} = \frac{2y - 3 - 1}{(3-1)} = \frac{2y - 4}{2}$$

$$\boxed{y = \eta + 2}$$

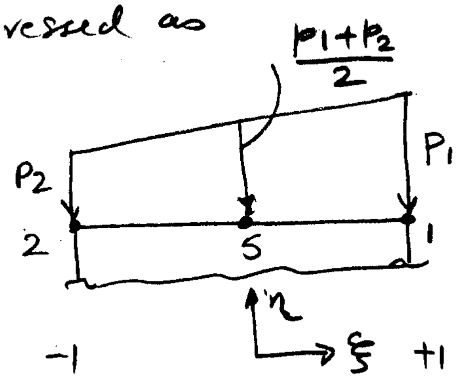
$$\frac{\partial x}{\partial \xi} = 1 \quad , \quad \frac{\partial x}{\partial \eta} = 0$$

$$\frac{\partial x}{\partial \eta} = 0 \quad , \quad \frac{\partial y}{\partial \eta} = 1$$

$$\Rightarrow \quad J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{Det}\,|J| = 1$$

Example Problem

The pressure variation can be expressed as



$$f_\eta = \frac{P_1+P_2}{2} + \frac{P_1-P_2}{2}\,\xi$$

Work Equivalent Nodal Forces at Nodes 1, 2, & 5 are found as follows

$$\{P\} = \int_{-1}^{+1} t \begin{bmatrix} N_1 & 0 \\ N_2 & 0 \\ \vdots & \vdots \\ N_5 & 0 \\ \vdots & \vdots \\ N_8 & 0 \\ \hline 0 & N_1 \\ 0 & N_2 \\ \vdots & \vdots \\ & N_5 \\ \vdots & \vdots \\ 0 & N_8 \end{bmatrix} \begin{Bmatrix} 0 \\ \frac{P_1+P_2}{2} + \frac{P_1-P_2}{2}\xi \end{Bmatrix} d\xi$$

The above integral need to be evaluated on edge $1 \to 5 - 2$ on which all N_i would be $= 0$ except for N_1, N_5, N_2

On edge $1 \to 5 \to 2$ $\eta = 1$ so

$$N_1(\xi,\eta) = N_1(\xi,\eta=1) = \frac{1}{4}(1+\xi)(1+\eta) - \frac{1}{2}(N_6 + \overset{0}{N_7})$$

$$= \frac{1}{4}(1+\xi)(2) - \frac{1}{2}\left(\frac{1}{2}(1-\xi^2)\,2\right)$$

$$= \frac{1}{2}(1+\xi) - \frac{1}{2}(1-\xi^2)$$

$$= \frac{1}{2}(1+\xi)(1-(1-\xi))$$

$$= \frac{1}{2}\xi(1+\xi)$$

$$N_2 = \frac{1}{4}(1-\xi)(1+\eta) - \frac{1}{2}\left(N_7 + \overset{\nearrow 0}{N_8}\right)$$

$$= \frac{1}{4}(1-\xi)(2) - \frac{1}{2}\left(\frac{1}{2}(1-\xi^2)(2)\right)$$

$$= \frac{1}{2}(1-\xi) - \frac{1}{2}(1-\xi^2)$$

$$= \frac{1}{2}(1-\xi)\left(1-(1+\xi)\right)$$

$$N_2 = -\frac{1}{2}\xi(1-\xi)$$

$$N_5 \equiv N_7 = \frac{1}{2}(1-\xi^2)(1+\eta) = \frac{1}{2}(1-\xi^2)(1+1)$$

$$N_5 = (1-\xi^2)$$

Hence the vertical Dir Nodal Loads for nodes 1, 2 & 5 are

$$\begin{Bmatrix} P_{1y} \\ P_{2y} \\ P_{5y} \end{Bmatrix} = \int_{-1}^{+1} t \begin{bmatrix} N_1 \\ N_2 \\ N_5 \end{bmatrix} \left[\frac{p_1+p_2}{2} + \frac{p_1-p_2}{2}\xi \right] d\xi$$

$$= t\int_{-1}^{+1} \begin{bmatrix} \frac{1}{2}\xi(1+\xi) \\ -\frac{1}{2}\xi(1-\xi) \\ (1-\xi^2) \end{bmatrix} \left[\frac{p_1+p_2}{2} + \frac{p_1-p_2}{2}\xi \right] d\xi$$

Example Problem

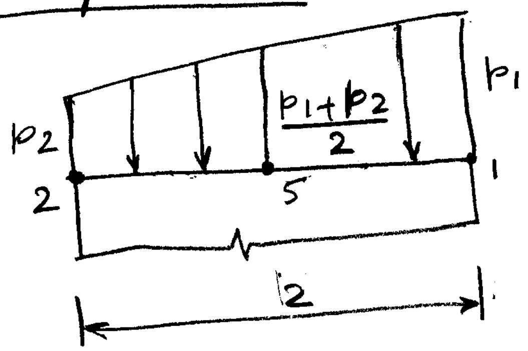
$$\begin{Bmatrix} P_{1y}\\ P_{2y}\\ P_{5y}\end{Bmatrix} = t\int_{-1}^{+1} \begin{Bmatrix} \frac{p_1+p_2}{4}\left(\xi+\xi^2\right) + \frac{p_1-p_2}{4}\left(\xi^2+\xi^3\right) \\ \frac{p_1+p_2}{4}\left(-\xi+\xi^2\right)+\frac{p_1-p_2}{4}\left(-\xi^2+\xi^3\right)\\ \frac{p_1+p_2}{2}\left(1-\xi^2\right)+\frac{p_1-p_2}{2}\left(\xi-\xi^3\right)\end{Bmatrix} d\xi$$

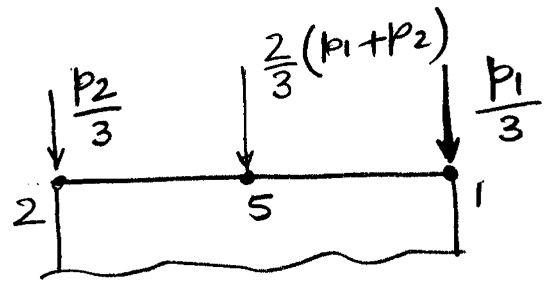
$$= t \left.\begin{bmatrix} \frac{\xi^2}{2}+\frac{\xi^3}{3} & \frac{\xi^3}{3}+\frac{\xi^4}{4} \\ -\frac{\xi^2}{2}+\frac{\xi^3}{3} & -\frac{\xi^3}{3}+\frac{\xi^4}{4} \\ 2\left(\xi-\frac{\xi^3}{3}\right) & 2\left(\frac{\xi^2}{2}-\frac{\xi^4}{4}\right)\end{bmatrix}\right|_{-1}^{+1} \begin{bmatrix} \frac{p_1+p_2}{4} \\ \frac{p_1-p_2}{4}\end{bmatrix}$$

$$= t \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} \\ 2\left(2-\frac{2}{3}\right) & 0 \end{bmatrix} \begin{Bmatrix} \frac{p_1+p_2}{4} \\ \frac{p_1-p_2}{4}\end{Bmatrix}$$

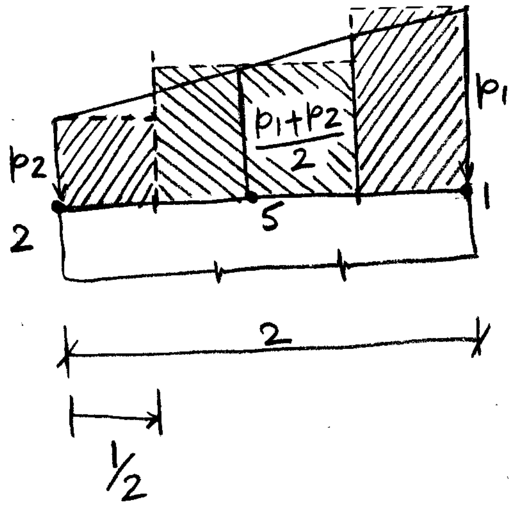
$$= t \begin{Bmatrix} \frac{2(p_1+p_2)}{12} + \frac{2(p_1-p_2)}{12} \\ \frac{2(p_1+p_2)}{12} - \frac{2(p_1-p_2)}{12} \\ \frac{8}{12}(p_1+p_2) \end{Bmatrix} = t \begin{Bmatrix} \frac{1}{3}p_1 \\ \frac{1}{3}p_2 \\ \frac{2}{3}(p_1+p_2)\end{Bmatrix}$$

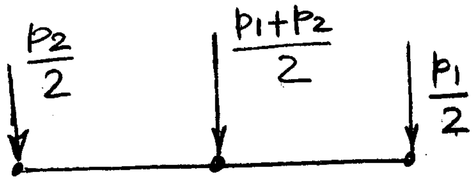
Example Problem





WORK EQUIVALENT
NODAL LOADS





NODAL LOADS
BASED ON TRIBUTARY AREAS

Total Pressure Force $= \frac{p_1+p_2}{2} \times 2 = p_1+p_2$

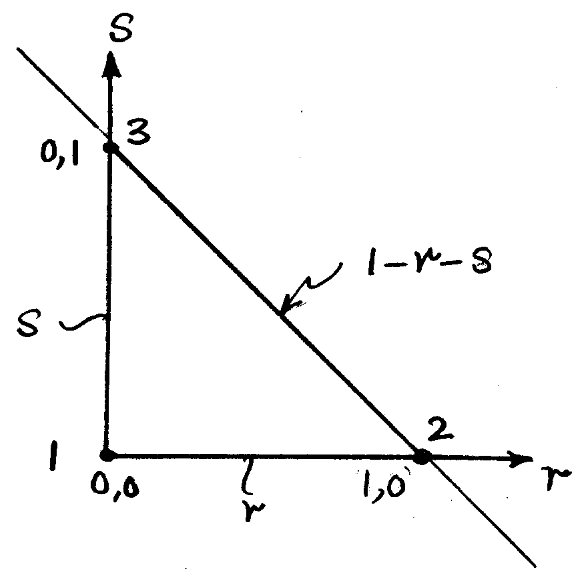
Sum Work Equiv. Nodal Forces $= \frac{p_2}{3} + \frac{2}{3}(p_1+p_2) + \frac{p_1}{3} = p_1+p_2$

Sum Tributary Area Est. Loads $= \frac{p_2}{2} + \frac{p_1+p_2}{2} + \frac{p_1}{2} = p_1+p_2$

Both the work equivalent nodal loads and the tributary area based nodal loads satisfy equilibrium.

ISOPARAMETRIC TRIANGULAR ELEMENTS

3-Noded Triangle



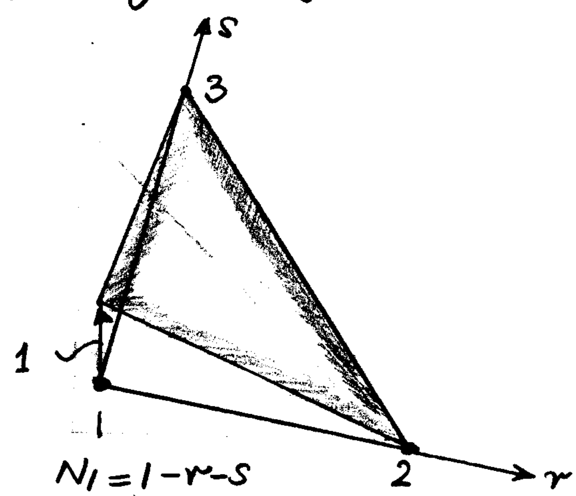
Shape Function for node 1 can be obtained by the equation of line passing through Nodes 2-3

Thus

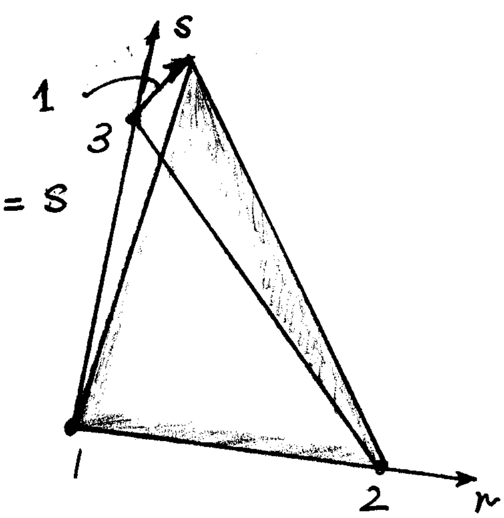
$N_1 = 1 - r - s$

$N_2 = r$

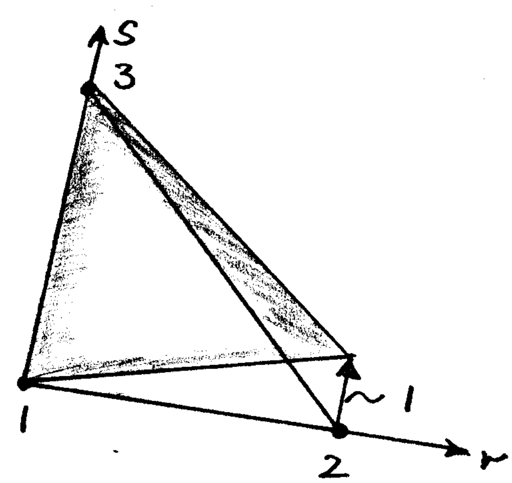
$N_3 = s$



$N_1 = 1 - r - s$



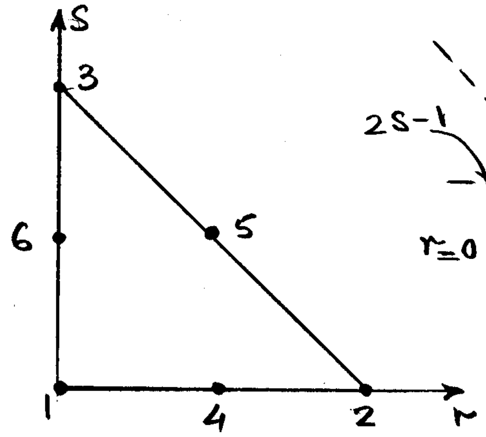
$N_3 = s$

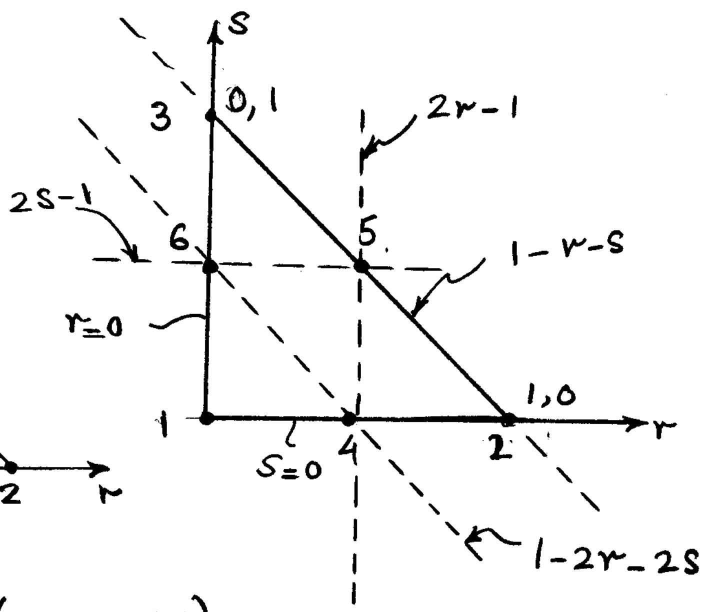


$N_2 = r$

ISOPARAMETRIC TRIANGULAR ELEMENTS

6-Noded Triangle





$$N_1 = (1-r-s)(1-2r-2s)$$

$$N_2 = r(2r-1)$$

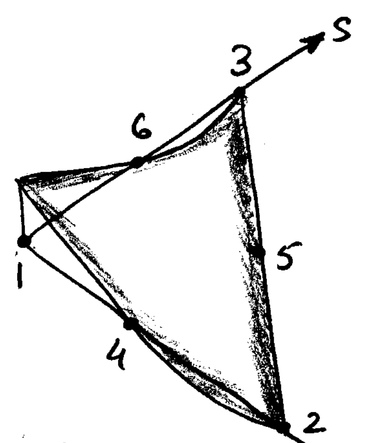
$$N_3 = s(2s-1)$$

$$N_4 = r(1-r-s) \qquad = \tfrac{1}{2}\left(1-\tfrac{1}{2}-0\right) = \tfrac{1}{4} \text{ @ } N_1$$

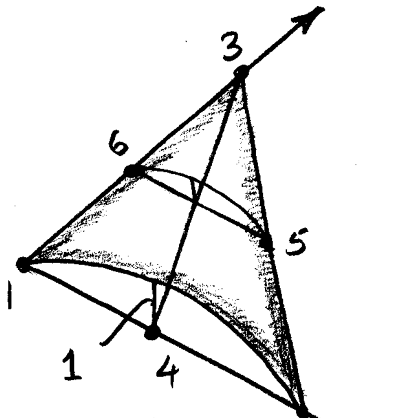
$$\Rightarrow N_4 = 4r(1-r-s)$$

$$N_5 = 4rs$$

$$N_6 = 4s(1-r-s)$$



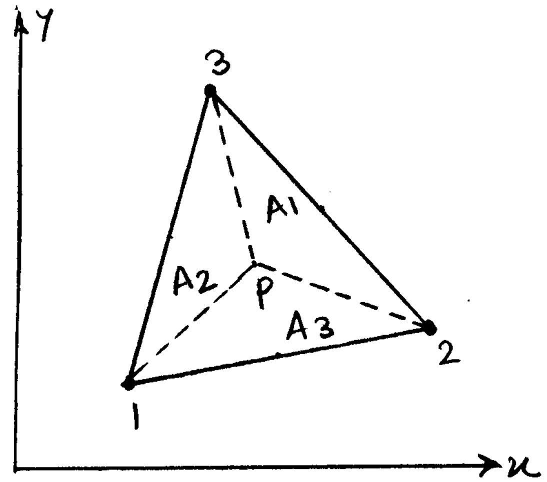
N_1



N_4

TRIANGULAR ELEMENTS
AREA COORDINATES

For Triangular Elements if their edges are straight and the nodes are uniformly spaced it is possible to write their shape functions using "Area Coordinates". Furthermore, it is possible in such case to evaluate the various integrals required for Finite Element Solution using analytical Formulas as the Jacobian Matrix for the mapping is constant.



The position of any Pt "P" within the triangle can be described in terms of area coordinates. The Pt "P" divides the triangle into 3 sub-areas the ratios of which to the total area of the triangle are called "Area Coordinates"

Area Coordinates of Pt "P"

$$L_1 = \frac{A_1}{A}, \quad L_2 = \frac{A_2}{A}, \quad L_3 = \frac{A_3}{A}$$

A = Total area of Triangle

$$L_1 + L_2 + L_3 = \frac{A_1}{A} + \frac{A_2}{A} + \frac{A_3}{A} = 1$$

Triangular Elements & Area Coordinates

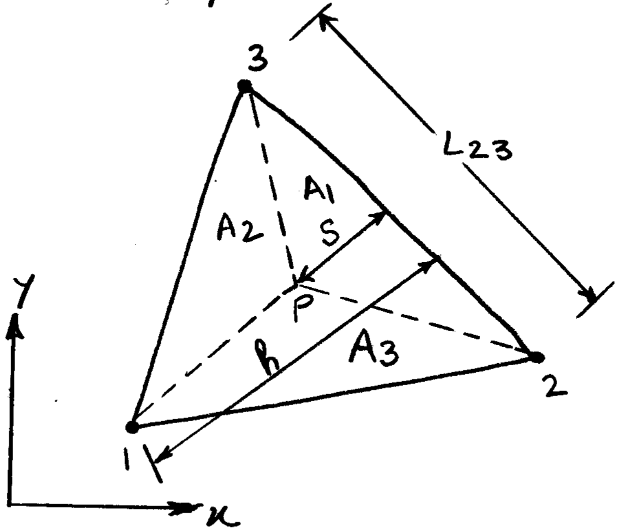
The centroid of a straight sided triangle is located at area Coordinates

$$L_1 = L_2 = L_3 = \frac{1}{3}$$

in Cartesian Coordinates the centroid is located at

$$\frac{x_1 + x_2 + x_3}{3} \quad \text{and} \quad \frac{y_1 + y_2 + y_3}{3}$$

Area Coordinates can also expressed as ratios of lengths



$$A_1 = \frac{L_{23}\,S}{2}$$

$$A = \frac{L_{23}\,h}{2}$$

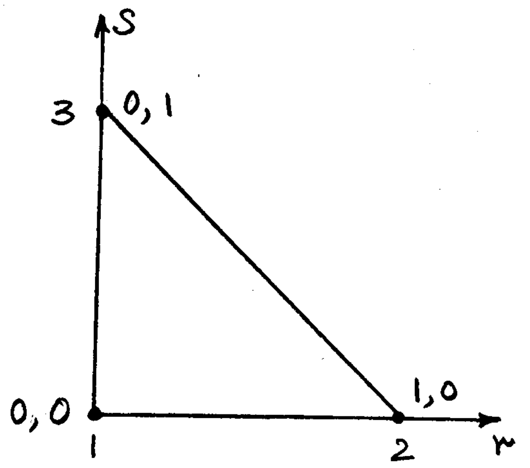
$$L_1 = \frac{A_1}{A} = \frac{L_{23}\,S}{2} \times \frac{2}{L_{23}\,h}$$

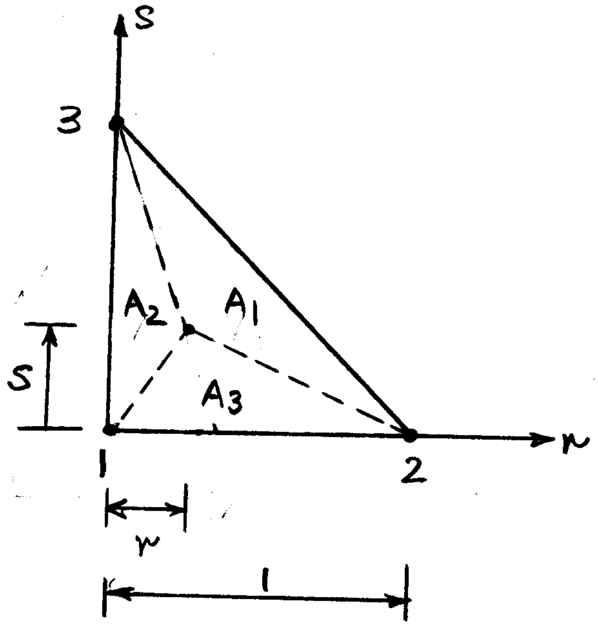
$$\boxed{L_1 = \frac{A_1}{A} = \frac{S}{h}}$$

Triangular Elements
& Area Coordinates

Relation Between Area Coords & Isoparametric Coords

The relation between Area Coordinates L_i and Isoparametric Shape Functions for triangular elements is easily establishable if we consider Area coordinates as ratios of lengths.





$$A_2 = \frac{1 \cdot r}{2} = \frac{r}{2}$$

$$A_3 = \frac{1 \cdot s}{2} = \frac{s}{2}$$

$$A = \frac{1}{2}$$

$$L_2 = \frac{A_2}{A} = \frac{\frac{r}{2}}{\frac{1}{2}} = r.$$

$$L_3 = \frac{A_3}{A} = \frac{s/2}{1/2} = s$$

$$L_1 = 1 - L_2 - L_3 = 1 - r - s$$

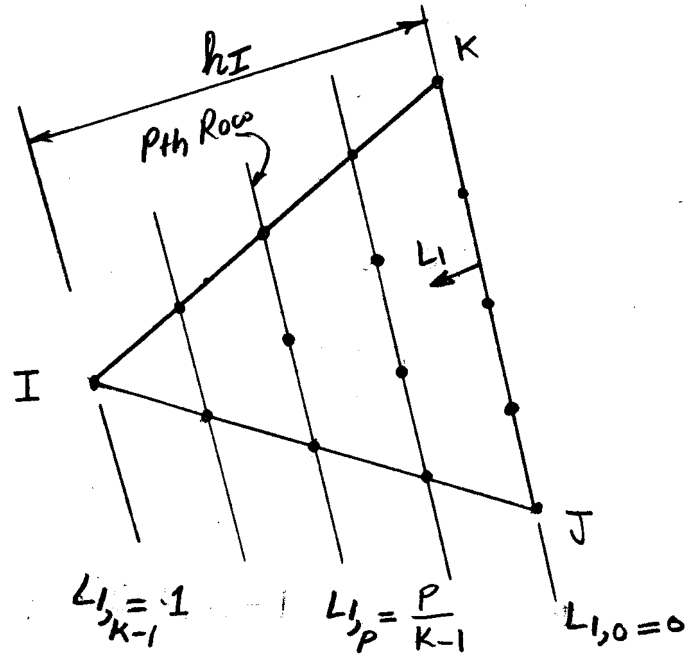
In this case the area coordinates are equal to shape Functions in parametric coordinates.

$$L_1 = N_1(r,s)$$

$$L_2 = N_2(r,s)$$

$$L_3 = N_3(r,s)$$

Area Coordinates for Higher Order Triangular Elements



Consider the higher order triangular element shown above containing "K" number of nodes per side that are equally spaced. Then the total number of nodes in the element is:

$$n = \sum_{i=0}^{K-1} (K-i) = K + (K-1) + (K-2) \cdots + 1 = \frac{1}{2} K(K+1)$$

The degree of interpolation Function is $K-1$

The Corner nodes are denoted by I, J and K

$h_I =$ Perpendicular distance of node I from side JK

If $L_{1,0} = 0 = L_1$ Coord of line JK

$L_{1,K-1} = 1 = L_1$ Coord of Corner Node I

Then if nodes are equally spaced

$$L_{1,p} = L_1 \text{ Coord of Pth Row from } JK = \frac{P}{K-1}$$

Area Coordinates for
Higher Order Triangular Elements

The interpolation Shape Function for the corner node I should be zero everywhere except @ I $L_{1,k-1}$

Using the Lagrangian Interpolation formula we can write

$$Na_I = \prod_{\substack{p=0 \\ p \neq k-1}}^{k-1} \frac{L_1 - L_{1,p}}{L_I - L_{1,p}}$$

$$Na_I = \frac{(L_1 - L_{1,0})(L_1 - L_{1,1}) \cdots (L_1 - L_{1,k-2})}{(L_I - L_{1,0})(L_I - L_{1,1}) \cdots (L_I - L_{1,k-2})}$$

Area Coordinates For Triangular Elements

Simillarly

$$N_{a3} = L_3(2L_3 - 1)$$

$$N_{a4} = L_1 L_2$$

$$N_{a4}\Big|_{@\text{Node } 4} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$\Rightarrow N_{a4} = 4 L_1 L_2$$

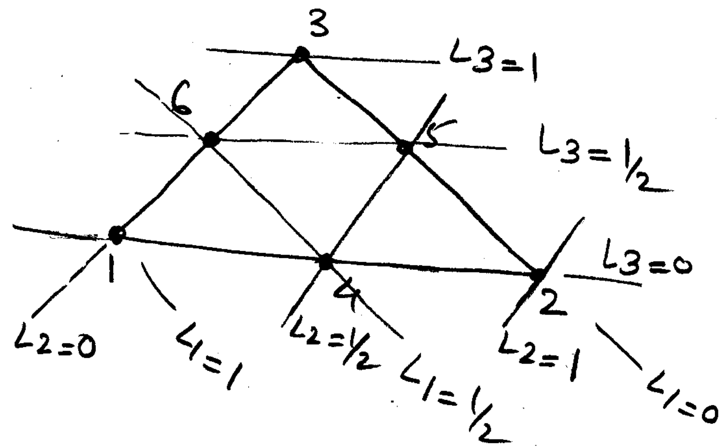
Simillarly

$$N_{a5} = 4 L_2 L_3$$

$$N_{a6} = 4 L_1 L_3$$

<u>Summarizing</u> The shape Function in area coordinates are

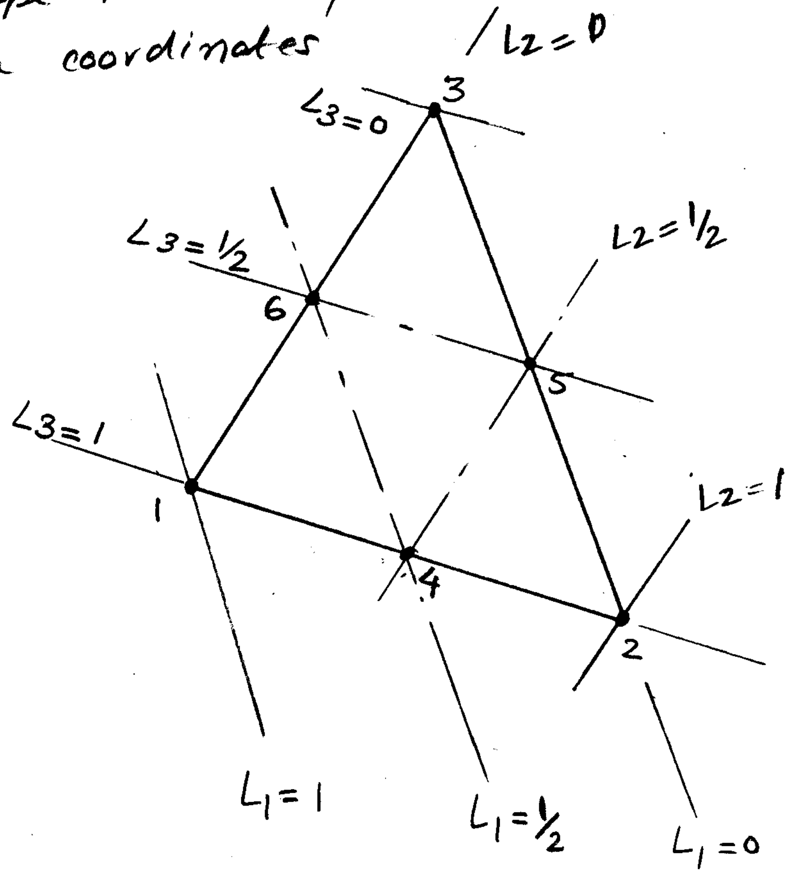
$$\boxed{\begin{aligned} N_{a1} &= L_1(2L_1 - 1) \\ N_{a2} &= L_2(2L_2 - 1) \\ N_{a3} &= L_3(2L_3 - 1) \\ N_{a4} &= 4 L_1 L_2 \\ N_{a5} &= 4 L_2 L_3 \\ N_{a6} &= 4 L_1 L_3 \end{aligned}}$$



Area Coordinates for Triangular Elements

Example Problem

Find the shape Functions for the Six Noded Triangle in area coordinates



$$Na_1 = \prod_{\substack{p=0 \\ p \neq k}}^{K} \frac{L_1 - L_{1,p}}{L_{1,1} - L_{1,p}}$$

$$Na_1 = \frac{(L_1 - 0)(L_1 - \frac{1}{2})}{(1-0)(1-\frac{1}{2})} = \frac{2L_1(2L_1 - 1)}{2(1)(2-1)} = L_1(2L_1 - 1)$$

$$Na_1 \Big|_{@\text{Node}1} = 1(2-1) = 1 \quad \text{Ok}$$

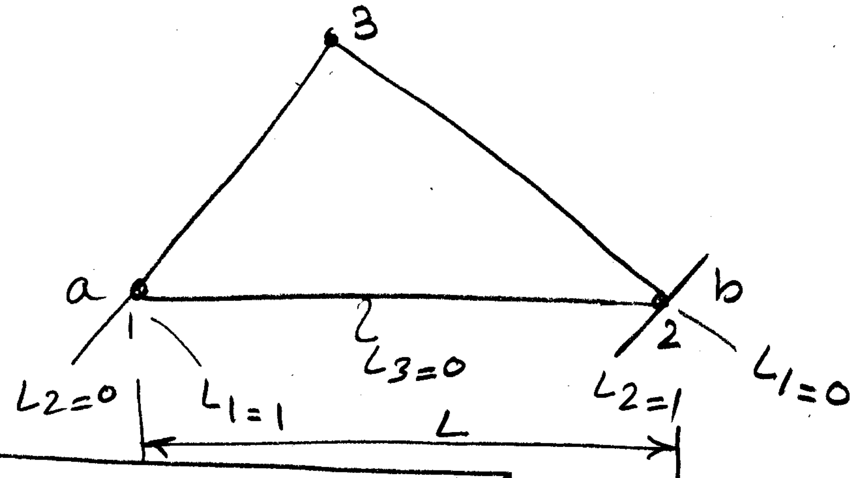
$$Na_2 = \prod_{\substack{p=0 \\ p \neq k}}^{K} \frac{L_2 - L_{2,p}}{L_{2,2} - L_{2,p}}$$

$$Na_2 = \frac{(L_2 - 0)(L_2 - \frac{1}{2})}{(1-0)(1-\frac{1}{2})} = L_2(2L_2 - 1)$$

Area Coordinates For Triangular Elements

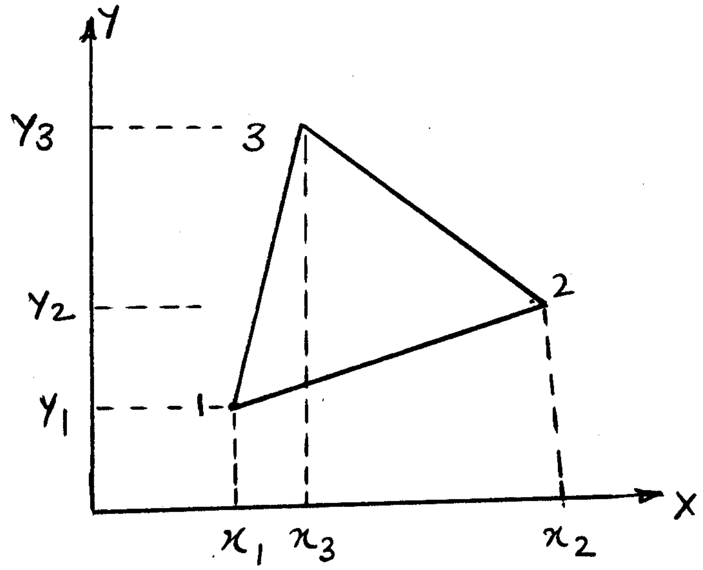
Integration Formulas:

Usage of area coordinates facilitates integration of shape functions over line paths and areas.



$$\boxed{\begin{aligned}\int_a^b L_1^k L_2^l \, dL &= \frac{k!\,l!}{(1+k+l)!} \cdot L \\ \int_A L_1^k L_2^l L_3^m \, dA &= 2A \frac{k!\,l!\,m!}{(2+k+l+m)}\end{aligned}}$$

Area Coordinates For Triangular Elements



The area of the above Triangle is equal to

$$A = \text{Area } x_1\,1\,3\,x_3 + \text{Area } x_3\,3\,2\,x_2 - \text{Area } x_1\,1\,2\,x_2$$

$$= (x_3 - x_1)\left(\frac{y_3 + y_1}{2}\right) + (x_2 - x_3)\left(\frac{y_3 + y_2}{2}\right)$$

$$- (x_2 - x_1)\left(\frac{y_2 + y_1}{2}\right)$$

$$\Rightarrow \quad 2A = y_1(x_3 - x_1 - x_2 + x_1) + y_2(x_2 - x_3 - x_2 + x_1)$$

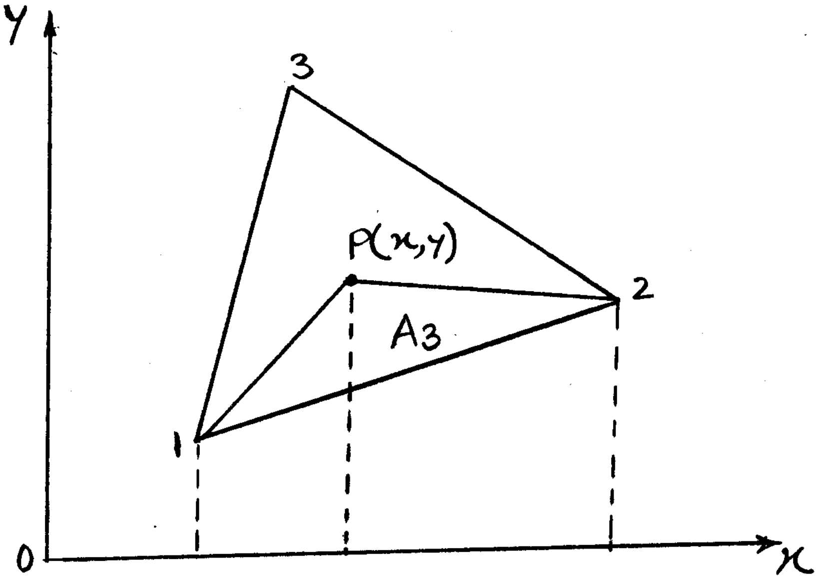
$$+ y_3(x_3 - x_1 + x_2 - x_3)$$

$$2A = y_1(x_3 - x_2) + y_2(x_1 - x_3) + y_3(x_2 - x_1)$$

$$\Rightarrow \quad 2A = \text{Det} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

Area Coordinates For Triangular Elements

Now the area of any of the subtriangles A_1, A_2, A_3 can be found in a simillar way



$$2A_3 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{vmatrix} = 1(x_1y_2 - x_2y_1) + x(y_1 - y_2) + y(x_2 - x_1)$$

$$x_1y_2 - x_2y_1 = \mathrm{Det}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} =$$ Twice the area of Triangle formed by Origin and pts 1 & 2

Thus we can write

$$2A_3 = 2A_{12} + y_{12}x + x_{21}y$$

Or in general

$$\boxed{2A_i = 2A_{jk} + y_{jk}x + x_{kj}y}$$

$$\boxed{L_i = \frac{A_i}{A} = \frac{1}{2A}\left(2A_{jk} + y_{jk}x + x_{kj}y\right)}$$

Area Coordinates and Triangular Elements

Writing the previous expression for shape function in Area Coordinates in matrix form:

$$\boxed{\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2 y_3 - x_3 y_2 & y_2 - y_3 & x_3 - y_2 \\ x_3 y_1 - x_1 y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1 y_2 - x_2 y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}}$$

→ Area Coordinate Shape Functions in cartesian Coords.

Inverse of above relation is:

$$\boxed{\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix}}$$

The first of above equations is the identity

$$L_1 + L_2 + L_3 = 1$$

The centroid of the triangle is at $L_1 = L_2 = L_3 = 1/3$

$$x_c = \frac{1}{3}(x_1 + x_2 + x_3), \quad y_c = \frac{1}{3}(y_1 + y_2 + y_3)$$

Area Coordinates and Triangular Elements

Establishing relation between derivatives in area coordinates and cartesian coordinates.

By Chain Rule:

$$\frac{\partial f(L_1, L_2, L_3)}{\partial x} = \frac{\partial f}{\partial L_1}\frac{\partial L_1}{\partial x} + \frac{\partial f}{\partial L_2}\frac{\partial L_2}{\partial x} + \frac{\partial f}{\partial L_3}\frac{\partial L_3}{\partial x}$$

$$= \frac{1}{2A}\left[(y_2-y_3)\frac{\partial f}{\partial L_1} + (y_3-y_1)\frac{\partial f}{\partial L_2} + (y_1-y_2)\frac{\partial f}{\partial L_3}\right]$$

$$\frac{\partial f(L_1, L_2, L_3)}{\partial y} = \frac{\partial f}{\partial L_1}\frac{\partial L_1}{\partial y} + \frac{\partial f}{\partial L_2}\frac{\partial L_2}{\partial y} + \frac{\partial f}{\partial L_3}\frac{\partial L_3}{\partial y}$$

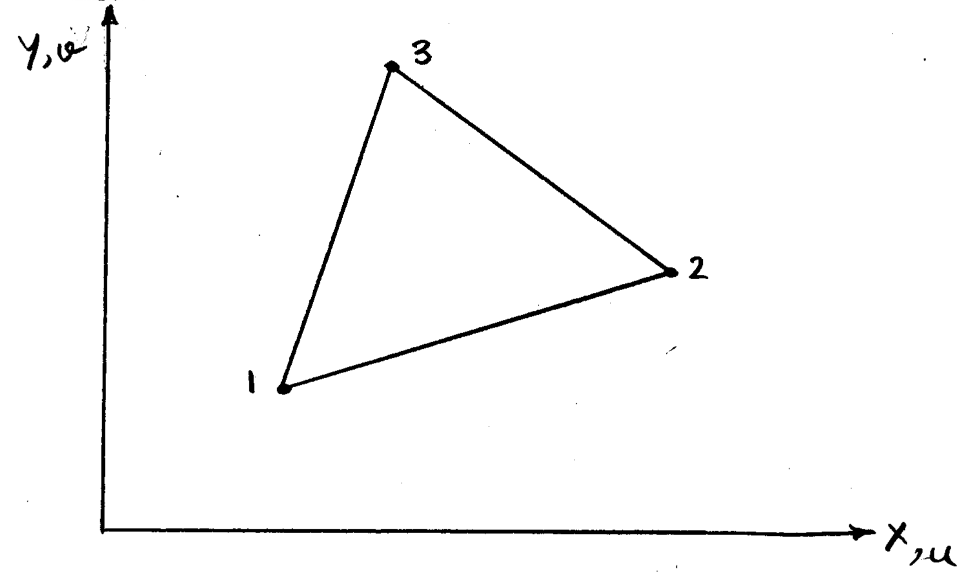
$$= \frac{1}{2A}\left[(x_3-x_2)\frac{\partial f}{\partial L_1} + (x_1-x_3)\frac{\partial f}{\partial L_2} + (x_2-x_1)\frac{\partial f}{\partial L_3}\right]$$

$$\begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} y_2-y_3 & y_3-y_1 & y_1-y_2 \\ x_3-x_2 & x_1-x_3 & x_2-x_1 \end{bmatrix}\begin{Bmatrix} \dfrac{\partial f}{\partial L_1} \\ \dfrac{\partial f}{\partial L_2} \\ \dfrac{\partial f}{\partial L_3} \end{Bmatrix}$$

$$\boxed{\begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix}\begin{Bmatrix} \dfrac{\partial f}{\partial L_1} \\ \dfrac{\partial f}{\partial L_2} \\ \dfrac{\partial f}{\partial L_3} \end{Bmatrix}}$$

Where $x_{32} = x_3 - x_2$ $\quad y_{23} = y_2 - y_3$ etc

Constant Strain Triangle CST
Using Area Coordinates



$$u = L_1 u_1 + L_2 u_2 + L_3 u_3$$

$$v = L_1 v_1 + L_2 v_2 + L_3 v_3$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} L_1 & L_2 & L_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_1 & L_2 & L_3 \end{bmatrix}_{2\times 6} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$u = N_a\, U$$

The Strains are given by:

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

Now making use of relation

$$\begin{Bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix} \begin{Bmatrix} \frac{\partial f}{\partial L_1} \\ \frac{\partial f}{\partial L_2} \\ \frac{\partial f}{\partial L_3} \end{Bmatrix}$$

Constant Strain Triangle CST Using Area Coordinates

$$\frac{\partial u}{\partial x} = \frac{1}{2A}\begin{bmatrix} y_{23} & y_{31} & y_{12}\end{bmatrix}\begin{Bmatrix} \frac{\partial u}{\partial L_1} \\ \frac{\partial u}{\partial L_2} \\ \frac{\partial u}{\partial L_3}\end{Bmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2A}\begin{bmatrix} y_{23} & y_{31} & y_{12}\end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3\end{Bmatrix}$$

$$\frac{\partial v}{\partial y} = \frac{1}{2A}\begin{bmatrix} x_{32} & x_{13} & x_{21}\end{bmatrix}\begin{Bmatrix} v_1 \\ v_2 \\ v_3\end{Bmatrix}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{2A}\left[\begin{array}{ccc|ccc} x_{32} & x_{13} & x_{21} & y_{23} & y_{31} & y_{12}\end{array}\right]\left\{\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ \hline v_1 \\ v_2 \\ v_3\end{array}\right\}$$

or in Matrix Form we can write

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy}\end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x}\end{bmatrix}\begin{Bmatrix} u \\ v\end{Bmatrix} = \underbrace{\frac{1}{2A}\left[\begin{array}{ccc|ccc} y_{23} & y_{31} & y_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{32} & x_{13} & x_{21} \\ x_{32} & x_{13} & x_{21} & y_{23} & y_{31} & y_{12}\end{array}\right]}_{B}\left\{\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ \hline v_1 \\ v_2 \\ v_3\end{array}\right\}$$

B STRAIN-DISPL MATRIX

Constant Strain Triangle Using Area Coordinates

We can also write:

$$\varepsilon_x = \frac{1}{2A} \sum_{i=1}^{n} b_i u_i$$

$$\varepsilon_y = \frac{1}{2A} \sum_{i=1}^{n} a_i v_i$$

$$\gamma_{xy} = \frac{1}{2A} \sum_{i=1}^{n} (a_i u_1 + b_i v_i)$$

Where,

$b_1, b_2, b_3 = Y_{23}, Y_{31}, Y_{12}$

$a_1, a_2, a_3 = X_{32}\ X_{13}\ X_{21}$

Now Element Stiffness Matrix

$$= K_e = \int_V B^T D B \, dv$$

$$= t \int_A B^T D B \, dA$$

$$K_e = \frac{t}{4A^2} \int_A \underbrace{\begin{bmatrix} Y_{23} & 0 & X_{32} \\ Y_{31} & 0 & X_{13} \\ Y_{12} & 0 & X_{21} \\ \hline 0 & X_{32} & Y_{23} \\ 0 & X_{13} & Y_{31} \\ 0 & X_{21} & Y_{12} \end{bmatrix}}_{6\times 3} \underbrace{\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ & E_{22} & E_{23} \\ \text{Sym.} & & E_{33} \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} Y_{23} & Y_{31} & Y_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & X_{32} & X_{13} & X_{21} \\ X_{32} & X_{13} & X_{21} & Y_{23} & Y_{31} & Y_{12} \end{bmatrix}}_{3\times 6} dA$$

Constant Strain Triangle Using Area Coordinates

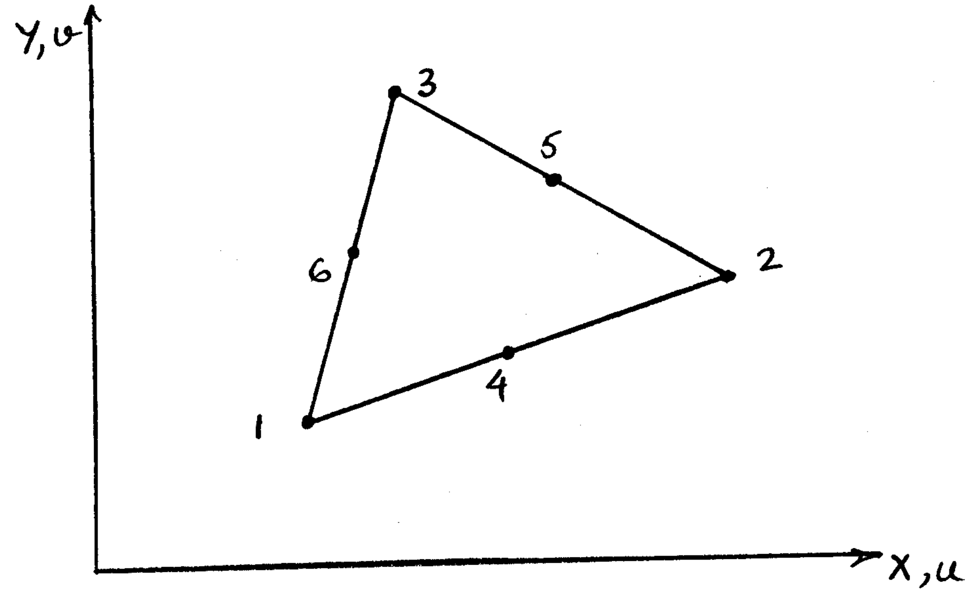
The previous expression for element stiffness matrix can be written as:

$$K_e = \left[\begin{array}{c|c} K_{uu_{3\times3}} & K_{uv_{3\times3}} \\ \hline K_{uv_{3\times3}}^T & K_{vv_{3\times3}} \end{array}\right]$$

$$\left.\begin{aligned} K_{uu,ij} &= E_{11}\, b_i b_j + E_{33}\, a_i a_j + E_{13}(b_i a_j + b_j a_i) \\ K_{vv,ij} &= E_{33}\, b_i b_j + E_{22}\, a_i a_j + E_{23}(b_i a_j + b_j a_i) \\ K_{uv,ij} &= E_{13}\, b_i b_j + E_{23}\, a_i a_j + E_{12}\, b_i a_j + E_{33}\, a_i b_j \end{aligned}\right\} \times \frac{t}{4A}$$

The above expression for element stiffness matrix yields the same/identical stiffness matrix obtained for CST element using Cartesion Coordinates.

LINEAR STRAIN TRIANGLE IN AREA COORDINATES



The Shape Function for 6-Noded LST have previously been derived and are summarized below for reference:

$$Na_1 = L_1(2L_1 - 1)$$

$$Na_2 = L_2(2L_2 - 1)$$

$$Na_3 = L_3(2L_3 - 1)$$

$$Na_4 = 4\,L_1 L_2$$

$$Na_5 = 4\,L_2 L_3$$

$$Na_6 = 4\,L_1 L_3$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \underbrace{\left[\begin{array}{cccccc|cccccc} N_1 & N_2 & N_3 & N_4 & N_5 & N_6 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & N_1 & N_2 & N_3 & N_4 & N_5 & N_6 \end{array}\right]}_{2\times 12} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ \hline V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{Bmatrix}$$

LINEAR STRAIN TRIANGLE IN AREA COORDINATES

$$\varepsilon_x = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

The above need to be converted to derivatives w.r.t area coordinates using the chain rule

$$\begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix} \begin{Bmatrix} \frac{\partial u}{\partial L_1} \\ \frac{\partial u}{\partial L_2} \\ \frac{\partial u}{\partial L_3} \end{Bmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2A} \begin{bmatrix} y_{23} & y_{31} & y_{12} \end{bmatrix} \begin{Bmatrix} \frac{\partial}{\partial L_1} \\ \frac{\partial}{\partial L_2} \\ \frac{\partial}{\partial L_3} \end{Bmatrix} \begin{bmatrix} N_1 \cdots N_6 & \vdots & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \end{Bmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{1}{2A} \underbrace{\begin{bmatrix} y_{23} & y_{31} & y_{12} \end{bmatrix}}_{Y} \underbrace{\begin{bmatrix} 4L_1-1 & 0 & 0 & 4L_2 & 0 & 4L_3 \\ 0 & 4L_2-1 & 0 & 4L_1 & 4L_3 & 0 \\ 0 & 0 & 4L_3-1 & 0 & 4L_2 & 4L_1 \end{bmatrix}_{3\times6}}_{\psi} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

LINEAR STRAIN TRIANGLE IN AREA COORDINATES

Simillarly,

$$\frac{\partial v}{\partial x} = \frac{1}{2A}\underbrace{\begin{bmatrix} x_{32} & x_{13} & x_{21} \end{bmatrix}}_{X} \begin{bmatrix} \psi \end{bmatrix}_{3\times6} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{Bmatrix}$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{2A}\begin{bmatrix} X & \vdots & Y \end{bmatrix}_{1\times6} \begin{bmatrix} \psi & 0 \\ 0 & \psi \end{bmatrix}_{6\times12} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ \hline v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{Bmatrix}$$

We Can Now write the Strain-Displacement Relation as:

$$\boxed{\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \underbrace{\frac{1}{2A}\begin{bmatrix} Y & 0 \\ 0 & X \\ X_{1\times3} & Y_{1\times3} \end{bmatrix}_{3\times6} \begin{bmatrix} \psi_{3\times6} & 0 \\ 0 & \psi_{3\times6} \end{bmatrix}_{6\times12}}_{[B]_{3\times12}} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_6 \\ \hline v_1 \\ v_2 \\ \vdots \\ v_6 \end{Bmatrix}}$$

$$\boxed{[B]_{3\times12} = \frac{1}{2A}\begin{bmatrix} Y & 0 \\ 0 & X \\ X_{1\times3} & Y_{1\times3} \end{bmatrix}_{3\times6} \begin{bmatrix} \psi_{3\times6} & 0 \\ 0 & \psi_{3\times6} \end{bmatrix}_{6\times12} \quad \text{STRAIN-DISPLACEMENT MATRIX}}$$

$$[B] = \frac{1}{2A}\begin{bmatrix} y_{23} & y_{31} & y_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{32} & x_{13} & x_{21} \\ x_{32} & 3_{13} & x_{21} & y_{23} & y_{31} & y_{12} \end{bmatrix}\begin{bmatrix} 4L_1-1 & 0 & 0 & 4L_2 & 0 & 4L_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4L_2-1 & 0 & 4L_1 & 4L_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4L_3-1 & 0 & 4L_2 & 4L_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4L_1-1 & 0 & 0 & 4L_2 & 0 & 4L_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4L_2-1 & 0 & 4L_1 & 4L_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4L_2-1 & 0 & 4L_2 & 4L_1 \end{bmatrix}$$

$$= \frac{1}{2A}\begin{bmatrix} y_{23}(4L_1-1) & y_{31}(4L_2-1) & y_{12}(4L_3-1) & 4(y_{23}L_2 + y_{31}L_1) & 4(y_{31}L_3 + y_{12}L_2) & 4(y_{23}L_3 + y_{12}L_1) & 0 & 0 \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & . & \ldots \\ x_{32}(4L_1-1) & x_{13}(4L_2-1) & x_{21}(4L_3-1) & 4(x_{32}L_2 + x_{13}L_1) & 4(x_{13}L_3 + x_{21}L_2) & 4(x_{32}L_3 + x_{21}L_1) & y_{23}(4L_1-1) & \ldots \end{bmatrix}$$

$$B^TDB = \begin{bmatrix} y_{23}(4L_1-1) & 0 & x_{32}(4L_1-1) \\ y_{31}(4L_2-1) & 0 & x_{13}(4L_2-1) \\ y_{12}(4L_3-1) & 0 & x_{21}(4L_3-1) \\ 4(y_{13}L_2 + y_{31}L_1) & 0 & 4(x_{32}L_2 + y_{13}L_1) \\ 4(y_{31}L_3 + y_{12}L_2) & 0 & 4(x_{13}L_3 + x_{21}L_2) \\ 4(y_{23}L_3 + y_{12}L_1) & 0 & 4(x_{32}L_3 + x_{21}L_1) \\ 0 & x_{32}(4L_1-1) & y_{23}(4L_1-1) \\ 0 & x_{13}(4L_2-1) & y_{31}(4L_2-1) \\ 0 & x_{21}(4L_3-1) & y_{12}(4L_3-1) \\ 0 & 4(x_{32}L_2 + x_{13}L_1) & 4(y_{23}L_2 + y_{31}L_1) \\ 0 & 4(x_{13}L_3 + x_{21}L_2) & 4(y_{31}L_3 + y_{12}L_2) \\ & 4(\ldots + \ldots L_1) & 4(y_{23}L_{\ldots} + y_{12}L_{\ldots}) \end{bmatrix}\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}\begin{bmatrix} y_{23}(4L_1-1) & y_{31}(4L_2-1) \ldots \\ 0 & 0 \ldots \\ x_{32}(4L_1-1) & x_{13}(4L_2-1) \ldots \end{bmatrix}$$

Carrying through these operations

$$
B^T D B = \frac{1}{4A^2}\begin{bmatrix}
(y_{23}^2 c_{11} + 2y_{23}x_{32}c_{13} + x_{32}^2 c_{33})(4L_1-1)^2 & (y_{23}y_{31}c_{11} + (y_{31}x_{32} + y_{23}x_{13})c_{13} + x_{13}x_{32}c_{33})(4L_1-1)(4L_2-1) \\
(y_{23}y_{31}c_{11} + (y_{23}x_{13} + x_{32}y_{31})c_{13} + x_{13}x_{32}c_{33})(4L_1-1)(4L_2-1) & (y_{31}^2 c_{11} + 2y_{31}x_{13}c_{13} + x_{13}^2 c_{33})(4L_2-1)^2 \\
\vdots & \vdots
\end{bmatrix}
$$

The resulting 12 x 12 matrix is then integrated term by term to obtain the desired stiffness matrix.

The process can be simplified, in terms of the integrations required, by using the concept of natural stiffness (i.e. computing stiffness with respect to generalized coordinates which have rigid body motions removed). In this particular case, we use the strains at the vertices of the triangle. Since the strains are linear,

$$
\varepsilon_x = L_1\,\varepsilon_{x_1} + L_2\,\varepsilon_{x_2} + L_3\,\varepsilon_{x_3} \quad \text{etc.} \qquad \varepsilon_{x_1} \equiv \varepsilon_x \text{ at node 1 etc.}
$$

where $\{\varepsilon_i\} = [\varepsilon_{x_1}\ \varepsilon_{x_2}\ \varepsilon_{x_3}\ \varepsilon_{y_1}\ \varepsilon_{y_2}\ \varepsilon_{y_3}\ \gamma_{xy_1}\ \gamma_{xy_2}\ \gamma_{xy_3}]^T$

This can be written more concisely as

$$
\{\varepsilon\} = \begin{bmatrix} \bar{N} & 0 & 0 \\ 0 & \bar{N} & 0 \\ 0 & 0 & \bar{N} \end{bmatrix} \{\varepsilon_i\}\ ; \quad [\bar{N}]_{1\times 3} = [L_1\ L_2\ L_3]
$$

The corner strains are now related to the nodal displacements by

$$\{\varepsilon_1\} = \frac{1}{2A}\left[\begin{array}{cc} b\psi_1 & 0 \\ b\psi_2 & 0 \\ b\psi_3 & 0 \\ \hdashline 0 & a\psi_1 \\ 0 & a\psi_2 \\ 0 & a\psi_3 \\ \hdashline a\psi_1 & b\psi_1 \\ a\psi_2 & b\psi_2 \\ a\psi_3 & b\psi_3 \end{array}\right]\{u\}$$

where

$[b] \equiv [y_{23}\ y_{31}\ y_{12}] = [b_1\ b_2\ b_3]$

$[a] = [x_{32}\ x_{13}\ x_{21}] = [a_1\ a_2\ a_3]$

and $[\psi_1]$ is $[\psi]$ evaluated at node 1, etc.

*{u} listed

$[u_1, \ldots u_6, v_1, \ldots v_6]^T$

or $\quad \{\varepsilon_1\} = [T]\{u\}$

$\qquad\qquad 9\text{x}12 \quad 12\text{x}1$

thus, $\quad \{\varepsilon\} = [\phi_\varepsilon][T]\{u\}$

$\qquad\qquad 3\text{x}9 \quad 9\text{x}12 \quad 12\text{x}1$

$\qquad [B]_{3\text{x}12}$

$$[k] = t[T]^T \int_A \underbrace{[\phi_\varepsilon]^T[D][\phi_\varepsilon]dA}[T]$$

This is the natural stiffness 9x9 (Three rigid body motions removed)

In general case

$$[D] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ & C_{22} & C_{23} \\ \text{Symm.} & & C_{33} \end{bmatrix}$$

$$[\phi_\varepsilon]^T [D] [\phi_\varepsilon] = \left[\begin{array}{c|c|c} C_{11} \; \overline{N}^T\overline{N} & C_{12} \; \overline{N}^T\overline{N} & C_{13} \; \overline{N}^T\overline{N} \\ & C_{22} \; \overline{N}^T\overline{N} & C_{23} \; \overline{N}^T\overline{N} \\ & & C_{33} \; \overline{N}^T\overline{N} \end{array}\right],$$

the only integration which must be performed is

$$\int_A \underset{3\times 3}{[\overline{N}^T\overline{N}]} \, dA$$

$$[\overline{N}]^T[\overline{N}] = \begin{bmatrix} L_1^2 & L_1L_2 & L_1L_3 \\ & L_2^2 & L_2L_3 \\ \text{Symm} & & L_3^2 \end{bmatrix} \qquad \int_A L_1^2 dA = \frac{A}{6} \qquad \int_A L_1L_2 dA = \frac{A}{12}$$

thus $\displaystyle\int_A [\phi_\varepsilon]^T [D] [\phi_\varepsilon] dA$

$$= \frac{A}{12} \left[\begin{array}{c|c|c} C_{11} \; Q & C_{12} \; Q & C_{13} \; Q \\ \hline & C_{22} \; Q & C_{33} \; Q \\ \hline \text{Symm} & & C_{33} \; Q \end{array}\right]; \quad \underset{3\times 3}{[Q]} \equiv \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Partition $[T]$ into $\dfrac{1}{2A} \begin{bmatrix} [U] & [0] \\ [0] & [V] \\ [V] & [U] \end{bmatrix} \qquad \underset{3\times 3}{[U], [V]}$

then $[k] = \frac{t}{48A} \begin{bmatrix} U^T & Q & V^T \\ 0 & V^T & U^T \end{bmatrix} \begin{bmatrix} C_{11}\,Q & C_{12}\,Q & C_{13}\,Q \\ & C_{22}\,Q & C_{23}\,Q \\ \text{Symm} & & C_{33}\,Q \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \\ V & U \end{bmatrix}$

this given on following page

$$[U] = \begin{bmatrix} b\psi_1 \\ b\psi_2 \\ b\psi_3 \end{bmatrix}_{3\times6} = \begin{bmatrix} 3b_1 & -b_2 & -b_3 & 4b_2 & 0 & 4b_3 \\ -b_1 & 3b_2 & -b_3 & 4b_1 & 4b_3 & 0 \\ -b_1 & -b_2 & 3b_3 & 0 & 4b_2 & 4b_1 \end{bmatrix}$$

$b_1 = y_{23}$
$b_2 = y_{31}$
$b_3 = y_{13}$

$a_1 = x_{32}$
$a_2 = x_{13}$
$a_3 = x_{21}$

[V] is obtained from [U] by substituting a_1 for b_1, a_2 for b_2, and a_3 for b_3.

The matrix product when multiplied out gives

$$[k_{uu}] = \frac{t}{48A} \quad [C_{11}\, U^TQU + C_{13}(V^TQU + U^TQV) + C_{33}V^TQV]$$

$$[k_{uv}] = \frac{t}{48A} \quad [C_{13}\, U^TQU + C_{12}U^TQV + C_{33}V^TQU + C_{23}V^TQV]$$

$$[k_{vv}] = \frac{t}{48A} \quad [C_{22}\, V^TQV + C_{23}\,(U^TQV + V^TQU) + C_{33}\, U^TQU]$$

$$[k] = \begin{bmatrix} [k_{uu}] & [k_{uv}] \\ \text{Symm.} & [k_{uv}] \end{bmatrix}$$

*order of listing

$\{u\} = u_1, \ldots u_6, V_1, \ldots V_6{}^T$

For isotropic material $C_{11} = C_{22} = \frac{E}{1-\nu^2}$, $C_{12} = \nu C_{11}$, $C_{33} = \frac{(1-\nu)}{2} C_{11}$

$$C_{13} = C_{23} = 0.$$

Stiffness matrix for LST element of constant thickness h :

$$
K = \frac{h}{48\,A}
\left[\begin{array}{ccc|ccc|ccc}
3b_1 & -b_1 & -b_1 & \cdot & \cdot & \cdot & 3a_1 & -a_1 & -a_1 \\
-b_2 & 3b_2 & -b_2 & \cdot & \cdot & \cdot & -a_2 & 3a_2 & -a_2 \\
-b_3 & -b_3 & 3b_3 & \cdot & \cdot & \cdot & -a_3 & -a_3 & 3a_3 \\
4b_2 & 4b_1 & \cdot & \cdot & \cdot & \cdot & 4a_2 & 4a_1 & \cdot \\
\cdot & 4b_3 & 4b_2 & \cdot & \cdot & \cdot & \cdot & 4a_3 & 4a_2 \\
4b_3 & \cdot & 4b_1 & \cdot & \cdot & \cdot & 4a_3 & \cdot & 4a_1 \\
\hline
\cdot & \cdot & \cdot & 3a_1 & -a_1 & -a_1 & 3b_1 & -b_1 & -b_1 \\
\cdot & \cdot & \cdot & -a_2 & 3a_2 & 3a_2 & -b_2 & 3b_2 & -b_2 \\
\cdot & \cdot & \cdot & -a_3 & -a_3 & 3a_3 & -b_3 & -b_3 & 3b_3 \\
\cdot & \cdot & \cdot & 4a_2 & 4a_1 & \cdot & 4b_2 & 4b_1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 4a_3 & 4a_2 & \cdot & 4b_3 & 4b_2 \\
\cdot & \cdot & \cdot & 4a_3 & \cdot & 4a_1 & 4b_3 & \cdot & 4b_1
\end{array}\right]
\left[\begin{array}{ccc|ccc|ccc}
2c_{11} & c_{11} & c_{11} & 2c_{12} & c_{12} & c_{12} & 2c_{13} & c_{13} & c_{13} \\
 & 2c_{11} & c_{11} & c_{12} & 2c_{12} & c_{12} & c_{13} & 2c_{13} & c_{13} \\
 & & 2c_{11} & c_{12} & c_{12} & 2c_{12} & c_{13} & c_{13} & 2c_{13} \\
\hline
 & & & 2c_{22} & c_{22} & c_{22} & 2c_{23} & c_{23} & c_{23} \\
 & & & & 2c_{22} & c_{22} & c_{23} & 2c_{23} & c_{23} \\
 & \text{SYMMETRIC} & & & & 2c_{22} & c_{23} & c_{23} & 2c_{23} \\
\hline
 & & & & & & 2c_{33} & c_{33} & c_{33} \\
 & & & & & & & 2c_{33} & c_{33} \\
 & & & & & & & & 2c_{33}
\end{array}\right]
\left[\begin{array}{cccccc|cccccc}
3b_1 & -b_2 & -b_3 & 4b_2 & \cdot & 4b_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-b_1 & 3b_2 & -b_3 & 4b_1 & 4b_3 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-b_1 & -b_2 & 3b_3 & \cdot & 4b_2 & 4b_1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 3a_1 & -a_2 & -a_3 & 4a_2 & \cdot & 4a_3 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -a_1 & 3a_2 & -a_3 & 4a_1 & 4a_3 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -a_1 & -a_2 & 3a_3 & \cdot & 4a_2 & 4a_1 \\
\hline
3a_1 & -a_2 & -a_3 & 4a_2 & \cdot & 4a_3 & 3b_1 & -b_2 & -b_3 & 4b_2 & \cdot & 4b_3 \\
-a_1 & 3a_2 & -a_3 & 4a_1 & 4a_3 & \cdot & -b_1 & 3b_2 & -b_3 & 4b_1 & 4b_3 & \cdot \\
-a_1 & -a_2 & 3a_3 & \cdot & 4a_2 & 4a_1 & -b_1 & -b_2 & 3b_3 & \cdot & 4b_2 & 4b_1
\end{array}\right]
\qquad \text{(III-29)}
$$

The nodal displacement vector is arranged as follows:

$$
r^T = \langle u_1 \;\; u_2 \;\; u_3 \;\; u_4 \;\; u_5 \;\; u_6 \;\vdots\; v_1 \;\; v_2 \;\; v_3 \;\; v_4 \;\; v_5 \;\; v_6 \rangle
$$

with isotropic material

$C_{11} = C_{22} = E/1-\nu^2$

$C_{12} = \nu E/1-\nu^2$

$C_{33} = (1-\nu)E/2(1-\nu^2)$

$C_{13} = C_{23} = 0$

$$[k_{uu}]_{6\times 6} = \frac{Et}{48A(1-\nu^2)} \left[U^TQU + \left(\frac{1-\nu}{2}\right) V^TQV\right]$$

$$[k_{uv}]_{6\times 6} = \frac{Et}{48A(1-\nu^2)} \left[\nu U^TQV + \left(\frac{1-\nu}{2}\right) V^TQU\right]$$

$$[k_{vv}]_{6\times 6} = \frac{Et}{48A(1-\nu^2)} \left[V^TQV + \left(\frac{1-\nu}{2}\right) U^TQU\right]$$

Only 3 matrices, U^TQU, V^TQV, U^TQV of relatively small order need to be evaluated.

Triangular element

$K_{11} = 12\, b_1^{\,2} + 6\,(1-\nu)\, a_1^{\,2}$

$K_{21} = \nu\, 12 a_1 b_1 + 6\,(1-\nu)\, a_1 b_1$

$K_{31} = -4 b_1 b_2 \;\; -2(1-\nu)\, a_1 a_2$

$K_{41} = -4\, a_2 b_1 \;\; -2(1-\nu)\, a_1 b_2$

$K_{51} = -4 b_1 b_3 \;\; -2(1-\nu)\, a_1 a_3$

$K_{61} = -4\,\nu\, a_3 b_1 \; -2(1-\nu) a_1 b_3$

$K_{71} = 16\, b_1 b_2 + 8(1-\nu)\, a_1 a_2$

$K_{81} = 16\, \nu a_2 b_1 + 8(1-\nu)\, a_1 b_2$

$K_{91} = 0$

$K_{101} = 16\,\nu\, a_3 b_1 + 8(1-\nu)\, a_1 b_2$

$K_{111} = 16\, b_1 b_3 + 8(1-\nu)\, a_1 a_3$

$K_{121} = 16\,\nu\, b_1 a_3 + 8(1-\nu) a_1 b_3$

$K_{22} = 12\, a_1^{\,2} + 6(1-\nu)\, b_1^{\,2}$

$K32 = 4\,\nu\, a_1 b_2 \;\; -2(1-\nu)\, b_1 a_2$

$K_{42} = -4\ a_1a_2\ \ -2(1-\nu)\ b_1b_2$

$K_{52} = -4\ \nu\ a_1b_3\ \ -2(1-\nu)\ b_1\ a_3$

$K_{62} = -4\ a_1\ a_3\ \ -2(1-\nu)\ b_1\ b_3$

$K_{72} = 16\nu\ a_1b_2\ \ +8(1-\nu)\ b_1\ a_2$

$K_{82} = 16\ a_1a_2\ \ +8(1-\nu)\ b_1b_2$

K92 $= 0$

$K_{102} = 0$

$K_{112} = 16\ \nu\ a_1b_3\ \ +8(1-\nu)\ b_1\ a_3$

$K_{122} = 16\ a_1a_3\ \ +8(1-\nu)\ b_1b_3$

$K_{33} = 12b_2^2 + 6(1-\nu)\ a_2^2$

$K_{43} = 12\nu\ a_2b_2 + 6(1-\nu)\ b_2a_2$

$K_{53} = -4\ b_2b_3\ \ -2(1-\nu)\ a_2a_3$

$K_{63} = -4\ \nu a_3b_2\ \ - 2(1-\nu)\ b_3a_2$

$K_{73} = 16\ b_1b_2\ \ +8(1-\nu)\ a_1a_2$

$K_{83} = 16\ \nu\ a_1b_2\ \ +8(1-\nu)\ b_1a_2$

$K_{93} = 16\ b_2b_3 +8(1-\ )\ a_2a_3$

$K_{103} = 16\ \nu\ a_3b_2 +8\ (1-\nu)\ b_3a_2$

$K_{113} = 0$

$K_{123} = 0$

$K_{44} = 12\ a_2^2\ \ +6(1-\nu)\ b_2^2$

$K_{54} = -4\ \nu\ a_2b_3\ \ -2(1-\nu)\ b_2a_3$

$$K_{64} = -4\ a_3a_2 \quad -2(1-\nu)\ b_3b_2$$

$$K_{74} = 16\,\nu\, a_2b_1 \quad +8(1-\nu)\ b_2a_1$$

$$K_{84} = 16\ a_1a_2 \quad +\ 8(1-\nu)\ b_1b_2$$

$$K_{94} \qquad 16\,\nu\, a_2b_3 \quad +8(1-\nu)\ b_2a_3$$

$$K_{104} = 16\ a_2a_3 \quad +8(1-\nu)\ b_2b_3$$

$$K_{114} = 0$$

$$K_{124} = 0$$

$$K_{55} = 12b_3^2 \quad +6(1-\ )\ a_3^2$$

$$K_{65} = 12\,\nu\, a_3b_3 \quad +6(1-\nu)\ b_3a_3$$

$$K_{75} = 0$$

$$K_{85} = 0$$

$$K_{95} = 16\ b_2b_3 \quad +8(1-\nu)\ a_2a_3$$

$$K_{105} = 16\nu\ a_2b_3 \quad +8(1-\nu)\ b_2a_3$$

$$K_{115} = 16\ b_1b_3 \quad +8(1-\nu)\ a_1a_3$$

$$K_{125} = 16\ \nu a_1b_3 \quad +\ 8(1-\nu)\ b_1a_3$$

$$K_{66} = 12a_3^2 \quad +6(1-\nu)\ b_3^2$$

$$K_{76} = 0$$

$$K_{86} = 0$$

$$K_{96} = 16\ \nu a_3b_2 \quad +8(1-\nu)\ b_3a_2$$

$$K_{106} = 16\ a_2a_3 \quad +8(1-\nu)\ b_2b_3$$

$$K_{116} = 16\nu b_1 a_3 + 8(1-\nu)\, a_1 b_3$$

$$K_{126} = 16\, a_1 a_3 + 8(1-\nu)\, b_1 b_3$$

$$K_{77} = 32\,(b_2^2 + b_1 b_2 + b_1^2) + (1-\nu)\, 16\,(a_2^2 + a_1 a_2 + a_1^2)$$

$$K_{87} = 16\nu(2a_2 b_2 + a_1 b_2 + a_2 b_1 + 2\, a_1 b_1) + 8(1-\nu)\,(2b_2 a_2 + b_1 a_2 + b_2 a_1 + 2b_1 a_1)$$

$$K_{97} = 16\,(b_2 b_3 + b_2^2 + 2b_1 b_3 + b_1 b_2) + 8(1-\nu)\,(a_2 a_3 + a_2^2 + 2a_1 a_3 + a_1 a_2)$$

$$K_{107} = 16\nu(b_2 a_3 + a_2 b_2 + 2b_1 a_3 + b_1 a_2) + 8(1-\nu)(a_2 b_3 + b_1 a_2 + 2a_1 b_3 + a_1 b_2)$$

$$K_{117} = 16\,(2b_2 b_3 + b_1 b_2 + b_1 b_3 + b_1^2) + 8(1-\nu)\,(2a_2 a_3 + a_1 a_2 + a_1 a_3 + a_1^2)$$

$$K_{127} = 16\,\nu(2a_3 b_2 + a_1 b_2 + a_3 b_1 + b_1 a_1) + 8(1-\nu)(2b_3 a_2 + b_1 a_2 + b_3 a_1 + a_1 b_1)$$

$$K_{88} = 32\,(a_2^2 + a_1 a_2 + a_1^2) + 16(1-\nu)(b_2^2 + b_1 b_2 + b_1^2)$$

$$K_{98} = 16\nu(a_2 b_3 + b_2 a_2 + 2a_1 b_3 + a_1 b_2) + 8(1-\nu)(b_2 a_3 + a_2 b_2 + 2b_1 a_3 + b_1 a_2)$$

$$K_{108} = 16(a_3 a_2 + a_2^2 + 2a_1 a_3 + a_1 a_2) + 8(1-\nu)\,(b_3 b_2 + b_2^2 + 2\, b_1 b_3 + b_1 b_2)$$

$$K_{118} = 16\nu(2a_2 b_3 + b_1 a_2 + a_1 b_3 + a_1 b_1) + 8(1-\nu)\,(2b_2 a_3 + a_1 b_2 + b_1 a_3 + b_1 a_1)$$

$$K_{128} = 16\,(2a_2 a_3 + a_1 a_2 + a_1 a_3 + a_1^2) + 8(1-\nu)\,(2b_2 b_3 + b_1 b_2 + b_1 b_3 + b_1^2)$$

$$K_{99} = 32\,(b_3^2 + b_1 b_3 + b_2^2) + 16(1-\nu)\,(a_3^2 + a_2 a_3 + a_1^2)$$

$$K_{109} = 16\nu(2a_3 b_3 + a_2 b_3 + a_3 b_2 + 2a_2 b_2) + 8(1-\nu)(2b_3 a_3 + b_2 a_3 + b_3 a_2 + 2b_2 a_2)$$

$$K_{119} = 16\,(b_3^2 + b_1 b_3 + b_2 b_3 + 2b_1 b_2) + 8(1-\nu)\,(a_3^2 + a_1 a_3 + 2a_1 a_3)$$

$$K_{129} = 16\nu(a_3 b_3 + a_1 b_3 + a_3 b_2 + 2a_1 b_2) + 8(1-\nu)\,(b_3 a_3 + b_1 a_3 + b_3 a_2 + 2b_1 a_2)$$

$$K_{1010} = 32(a_3^2 + a_2 a_3 + a_2^2) + 16(1-\nu)\,(b_3^2 + b_2 b_3 + b_2^2)$$

$$K_{1110} = 16\nu(a_3 b_3 + b_1 a_3 + a_2 b_3 + 2a_2 b_1) + 8(1-\nu)\,(b_3 a_3 + a_1 b_3 + b_2 a_3 + 2b_2 a_1)$$

$$K_{1210} = 16(a_3^2 + a_1 a_3 + a_2 a_3 + 2a_1 a_2) + 8(1-\nu)\,(b_3^2 + b_1 b_3 + b_2 b_3 + 2b_1 b_2)$$

$$K_{1211} = 16\nu(2_3b_3 + a_1b_3 + a_3b_1 + 2a_1b_1) + 8(1-\nu)(2b_3a_3 + b_1a_3 + b_3a_1 + 2b_1a_1)$$

$$K1212 = 32(a_3^2 + a_1a_3 + a_1^2) + 16(1-\nu)\ (b_3^2 + b_1b_3 + b_1^2)$$

The matrix is on the form

$$\begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ \\ \\ \end{bmatrix} \quad \begin{bmatrix} K_{11} & & & \cdots \\ K_{21} & K_{22} & & \cdots \\ K_{31} & K_{32} & K_{33} & \cdots \\ \vdots & \vdots & \vdots & \\ \end{bmatrix} \quad \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \\ \end{bmatrix} \quad \frac{Et}{48A(1-\nu^2)}$$

t = thickness

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