ERROR & CONVERGENCE IN FEM

SOURCES OF ERROR IN FEA SOLUTION

1. Modelling Errors

Arises from the difference between the actual physical problem and its mathematical idealization. The idealized FEA problem may be an oversimplification of the actual physical problem

- geometry may not be correctly represented
- material characterization may not be accurate
- The assumptions regarding the problem i.e whether plane strain, plane stress or axisymmetric may not be accurate.
- Boundary conditions may not be adequately reflected.

Discretization Errors

Refers to errors arising from inappropriate Finite Element meshing in posing the problem to be solved.

- Coarse or poor meshing
- Usage of inappropriate elements to analyze the posed problem

Truncation error or Roundoff Errors

Refers to loss of information arising from truncation of numbers and rounding-off of numbers as they are manipulated by the computer.

Truncation error is dependent upon the degree of precision used to store numbers for computation purposes.

If two numbers $x = 1.23456$ and $y = 1.23455$ are 6-digit representations of a number that actually has more than 6 digits, then some loss of accuracy has occured as the actual numbers are stored in the computer memory.

Error propagation occurs as difference of these number is taken:

$X - Y = 1 \times 10^{-5}$ is unreliable even in its single digit.

Manipulation Error

FE Analyses require solution of large system of simultaneous equations of form

$$[K]\{U\} = \{P\}$$

In general, solution techniques employed for solving the equations determine the solution $\{U\}$ within some tolerance limits, introducing minor errors.

In nonlinear analyses and time-dependent analyses analysis of next step uses results from previous step, resulting in accumulation of error as analysis progresses.

SOURCES OF ERROR

NUMERICAL ERROR

Is the combined effect of Truncation Error and Manipulation Error. To limit such errors calculations should be performed in "Double Precision" (each number = 16 bits) rather than in "Single Precision" (each number = 8 bits).

Numbers such as π and Gauss Pt wts should be stored with as much precision as possible.

USER GENERATED ERRORS

Refers to mistakes made by software user in posing the problem. such as using inappropriate element types, incorrect boundary conditions and inappropriate mesh generation.

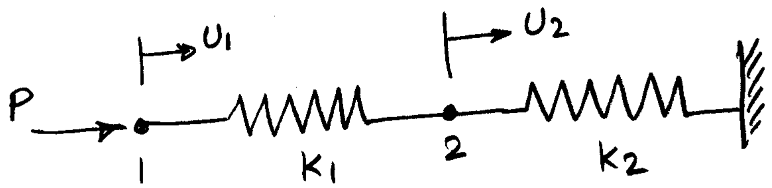
SOFTWARE BUGS

Possibility of programming errors (Bugs) in an FEA Software can not be completely eliminated. More versatile and complex the software, higher is the probability of bugs.

* "The most dangerous bug is one which does not halt program execution and goes unnoticed"

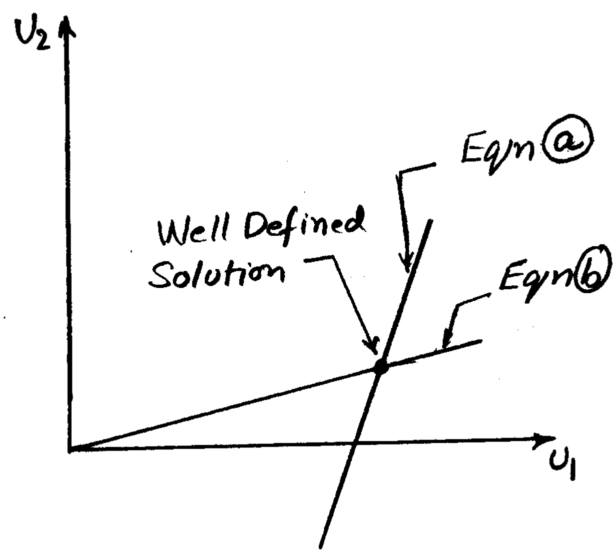
ILL-CONDITIONING

A set of linear simultaneous system of equations is considered to be ill-conditioned if the solution vector $\{U\}$ is sensitive to small changes in the coefficient Matrix (Stiffness Matrix).



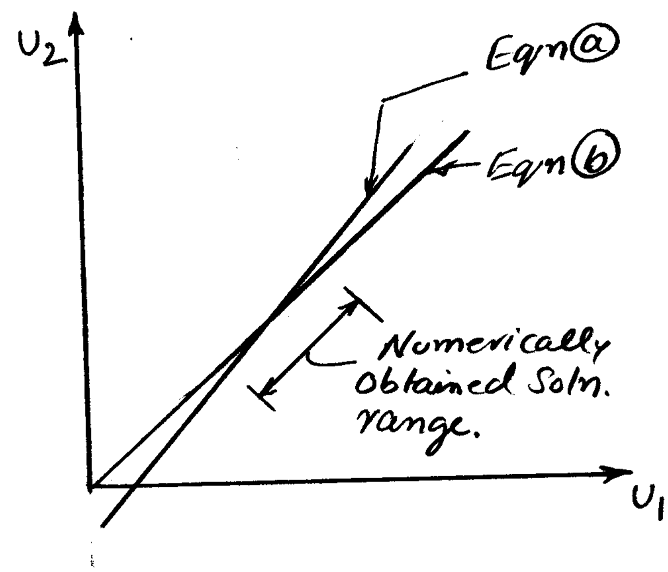
$$\begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1+K_2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix} \quad \begin{matrix} \text{——— (a)} \\ \text{——— (b)} \end{matrix}$$

Case $K_1 \ll K_2$
Well conditioned



Flexible Part Supported by Stiff Part

Case $K_1 \gg K_2$
Ill-Conditioned



Stiff Part Supported by Flexible Part.

ILL-CONDITIONING

Adding equation ⓐ to ⓑ we have

$$\left[(K_1 + K_2) - K_1\right] U_2 = P$$

which would yield

$$K_2 U_2 = P$$ if K_1 and K_2 are represented with infinite precision. However, within the confines of computer numeric operations following scenario may occor

Say $K_1 = 1.000{,}000$ $\quad K_2 = 4.444444 \times 10^{-6}$

with 6-digit precision we have

$$(K_1 + K_2) - K_1$$

$$= \left(1.000{,}000 + 0.000{,}00\text{----}\right) - 1.000{,}000$$

Truncation Loss

$$= 0.000{,}000 \quad = 4.0 \times 10^{-6}$$ with 6-digit Precision

Compared to

$$0.000{,}004{,}444{,}444 = 4.444{,}444 \times 10^{-6}$$

with 12-Digit Precision

$U_2 \longrightarrow \infty$ with 6-digit precision

Analogous to physical situation in which spring K_1 has no physical support or restraint and can undergo rigid body motion.

* Program will complain that stiffness matrix is "Singular"

ILL- CONDITIONED SYSTEMS

- Ill-Conditioning of Structural Systems often occurs when a stiff region is supported by much more flexible region

- Ill-Conditioned systems often have stiffness matrices in which the off-diagonal terms are relatively large compared to diagonal terms.

Condition Number

The suitability of a system of simultaneous linear system of equations to yield accurate and error free solution following matrix manipulation operations depends upon the "Condition Number" of the coefficient matrix (Stiffness Matrix)

If the system of equations is:

$$[K]\{U\} = \{P\}$$

Then Condition Number of $[K]$ is defined as

$$\text{Condition Number} = C(K) = \frac{\lambda_{max}}{\lambda_{min}}.$$

where

λ_{max} = highest Eigenvalue of $[K]$

λ_{min} = smallest Eigenvalue of $[K]$

Higher the Condition Number, higher the chances of inaccuracies in the solution vector $\{U\}$ obtained following matrix manipulations.

Error Estimation by Residual

If a system of equations such as one shown below has been solved

$$[K]\{U\} = \{R\}$$

yielding a solution vector $\{U\}$

Then we can calculate the Residual

$$\{\Delta R\} = \{R\} - [K]\{U\}$$

where $\{\Delta R\} \rightarrow \{0\}$ for an exact or accurate solution

and $\{\Delta R\} \neq \{0\}$ if inaccuracies are present

A scaler Norm of the error measure $\{\Delta R\}$ can be written as

$$\boxed{e = \frac{\{U\}^T \{\Delta R\}}{\{U\}^T \{R\}}}$$

e = ratio of work done by the residual loads to work done by actual loads as they act through displacement $\{U\}$.
Therefore "e" is an energy norm.

If $\{\Delta R\} \neq 0$. Iterative improvements can be carried out to the solution vector $\{U\}$ as follows:

$$\{\Delta R\}_i = \{R\} - [K]\{U\}_i$$

$$[K]\{\Delta U\}_i = \{\Delta R\}_i$$

$$\{U\}_{i+1} = \{U\}_i + \{\Delta U\}_i \xrightarrow{\text{Converged}} \{U\} = \{U\}_{i+1}$$

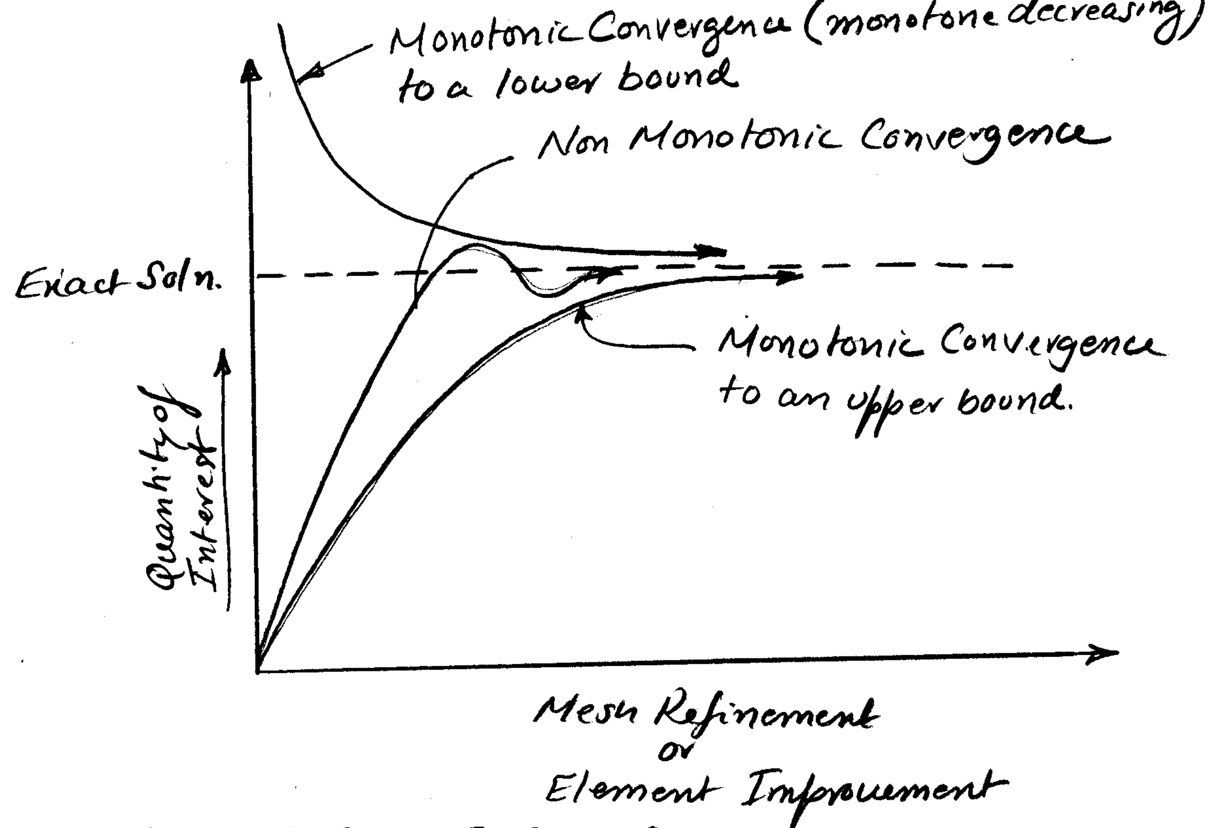
Not converged

Convergence of Finite Element Solution

The characteristic of a finite element solution to arrive at the exact solution of posed problem following mesh refinement, improvement of interpolation functions describing the displacement-field or by improved error minimization during solution of Finite Element equations, is called "convergence."

Monotonic Convergence

A Finite Element Analysis is said to be Monotonically Converging to an exact solution if each successive estimation of quantity of interest (displacement, stresses or strains) is greater than previous estimate but is yet bounded and is approaching in the limit to the exact solution.



DIFFERENT TYPES OF CONVERGENCES

Monotonic Convergence

The ideal convergence in Finite Element Analysis is the Monotonic Convergence with Monotone increasing and solution converging to an upper bound.

However, there are conditions and elements that are in use that result in convergence with monotone decreasing and convergence to a lower bound. Also usage of some elements results in Non-Monotonic Convergence.

Requirements for Monotonic Convergence with monotone increasing and solution converging to an upper bound.

For such Convergence reffered to as Monotonic Convergence, following 2 Conditions must be met:

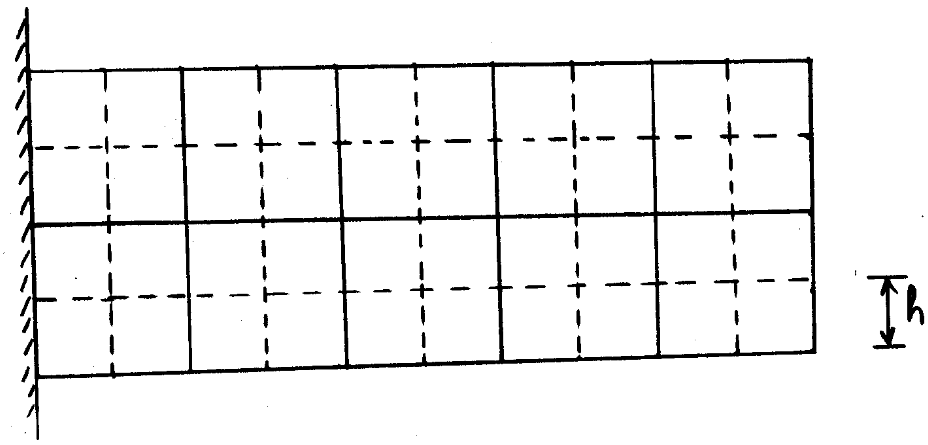
1. Completeness:- By this is meant Completeness of the element displacement field. i.e the displacement functions of the element must be able to represent the rigid body displacements/modes. and "constant strain states"

Monotonic Convergence

2. Compatibility: The requirement of compatibility means that the displacements within the element and accross the element boundaries must be continuous and that no gaps between elements occur when the element assemblage is loaded.

Types of Convergence and Rates of Convergence

h Refinement or h Convergence



One way of achieving convergence is to refine the mesh successively in such a manner that the previous mesh is contained in the successive refined meshes and the previous mesh lines are contained in the successive meshes

"h" is the mesh size parameter indicating the size of the element side or the diameter of a circle encompassing the typical element.

If "u" is the exact solution to the problem and "u_h" is the finite element solution. Then

$$\boxed{\|u - u_h\|_1 \leq c\, h^K}$$

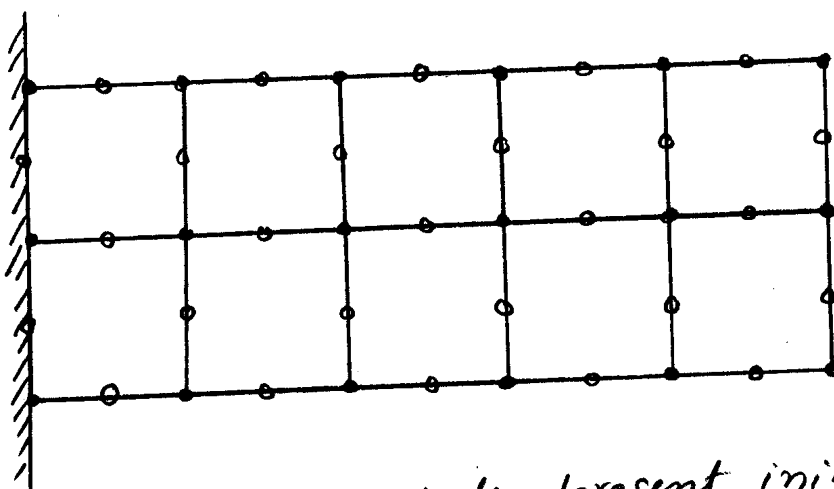
where c = constant independent of h but dependent upon material properties

K = Rate of convergence = order of the interpolating polynomial in element shape function.

The order of convergence in this case is "K" or equivalently we have $O(h^K)$ convergence.

p. Refinement or p Convergence

p refers to the highest complete polynomial in the element displacement field. p refinement or convergence consists of increasing p within elements without changing the number of elements. This is done by adding nodes to existing interelement boundaries.

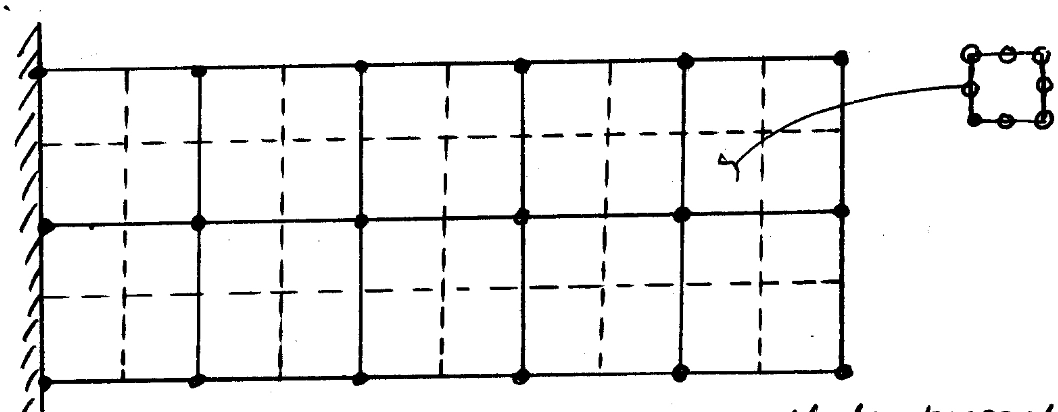


• — Nodes present initially

o — Nodes added for p refinement.

h/p Refinement or h/p Convergence

In h/p refinement or h/p convergence the number of elements is increased and at the same time the order of displacement field in the elements is increased.



[ ] — Elements added for h/p refinement

• — Nodes present initially

o — Nodes added in h/p refinement

h/p Refinement or h/p Convergence

Convergence in this case is at an exponential rate and has the form

$$\boxed{\|u - u_h\|_1 \leq \frac{c}{\exp\left[\beta (N)^{\gamma}\right]}} \quad \text{—— (1)}$$

c, β and γ = Constants

N = number of nodes in the mesh.

The h Refinement or convergence norm if written in above form is

$$\boxed{\|u - u_h\|_1 \leq \frac{c}{(N)^{K/d}}} \quad \text{—— (2)}$$

where, $d = 1, 2, 3$ respectively for 1, 2 and 3-D analyses

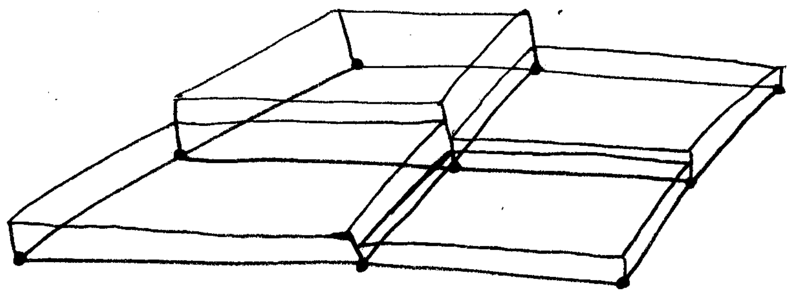
K = Rate of h convergence = Order of the interpolating polynomial in element Shape function.

Comparing (1) & (2) we conclude that h/p Refinement has a much faster rate of convergence as the rate of convergence in h/p refinement is exponential, whereas in h refinement it is algebraic.

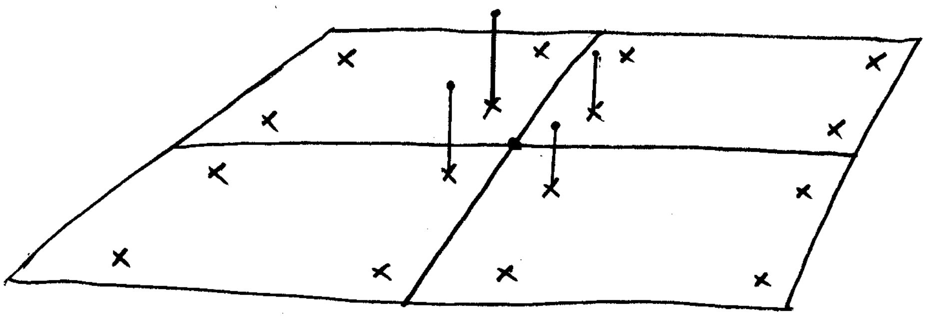
STRESS RECOVERY & SMOOTHING

The stresses at a common node between elements can be different when calculated by each of the adjoining element. Also, the stresses at Gauss pt locations in vicinity of a node can also be different.

* The discontinuity of stresses at nodes when estimated from adjoining elements provides a measure of the accuracy of analysis.



Plot of Element Stresses at a common Node showing Stress Stress Discontinuity.



Ordinates of stress magnitude ploted at Gauss pt locations showing different values at Gauss pts at a common Node

STRESS RECOVERY & SMOOTHING

Nomenclature,

$\{\varepsilon\}, \{\sigma\}$ — Strains and Stresses calculated element-by-element fashion; $\{\varepsilon\} = [B]\{U\}$

$$\{\sigma\} = [E]\{\varepsilon\}$$

$\{\overset{*}{\varepsilon}\}, \{\sigma^*\}$ — Strains and Stresses estimated by smoothing operation

σ, σ^* — One of the stresses in $\{\sigma\}$ or $\{\sigma^*\}$

Nodal Averaging Technique

$$\sigma^* = \frac{1}{n}\sum_{i=1}^{n} \sigma_i$$

Note:- One must avoid averaging accross physically valid discontinuities such as sudden change in material properties.

smoothed Stress variation over a single element is

$$\sigma^* = \lfloor N \rfloor \{\sigma_n^*\}$$

Vector of nodal averages of stress for the element.

STRESS SMOOTHING BY PATCH RECOVERY

In Patch Recovery or Stress Smoothing operations stresses are smoothed over cluster or a "patch" of elements

The smoothed stress can be expressed as:

$$\sigma^* = \lfloor P \rfloor \{a\} \quad \text{—— (1)}$$

$\lfloor P \rfloor$ Contains terms of polynomial used to describe variation of smoothed stresses over the patch.

$\{a\}$ = generalized coordinates to be determined.

$\{a\}$ can be determined by a least squares surface fitting technique subject to condition that the difference between smoothed stresses σ^* and element stresses σ (sampled at locations where they are most accurate, usually Gauss pts) is minimum.

$$Fp = \sum_{i=1}^{nsp} (\sigma^* - \sigma)_i^2 \quad , \quad nsp = \text{Number of sampling Pts in a patch.}$$

—— (2)

Substitution of Eqn (1) in (2) and minimization of Fp with respect to ai yields

$$[A]\{a\} = \{B\}$$

Where, $[A] = \sum_{i=1}^{nsp} \lfloor P \rfloor_i^T \lfloor P \rfloor_i$

$$[B] = \sum_{i=1}^{nsp} \lfloor P \rfloor_i^T \sigma_i$$

—— (3)

STRESS RECOVERY BY PATCH SMOOTHING

The generalized coordinates $\{a\}$ are determined by solving equation 3. It is preferable to have an overdetermined least squares surface-fit by sampling at more locations (preferably at all Gauss Pts in the patch) i.e

$$nsp > \text{Terms in } \{a\}$$

Note:

It has been observed that stresses determined by patch recovery method are super-convergent. Their convergence rate is atleast $O(h^{p+1})$. For Linear Elements Q4 and CST the convergence rate is $O(h^2)$; For Quadratic Elements Q8 and LST, the rate is $O(h^3)$ on the boundary and $O(h^4)$ for points internal to the mesh. The convergence rate of $O(h^4)$ is considered to be ultra-convergent.

Energy Based Error Norms

ZZ Error Estimate (By Zienkiewicz and Zhu)

If $\{\varepsilon\} = [B]\{U\}$ = Element by element strains

Then sum of strain energies of all the elements multiplied by 2 is defined as the square of the "global strain energy norm" $\|U\|^2$

$$\boxed{\|U\|^2 = \sum_{i=1}^{m} \int \{\varepsilon\}_i^T [E]_i \{\varepsilon\}_i \, dv} \quad \text{—} ①$$

m = number of elements in the region of interest.

Using the difference between $\{\varepsilon^*\}$ = smoothed strain-field and element by element strains $\{\varepsilon\}$, we can define the "Global Energy Error Norm" $\|e\|$

$$\boxed{\|e\|^2 = \sum_{i=1}^{m} \int \left(\{\varepsilon^*\}_i - \{\varepsilon\}_i\right)^T [E] \left(\{\varepsilon^*\}_i - \{\varepsilon\}_i\right) dv} \quad \text{—} ②$$

$\|e\|^2$ can be considered as an estimator of error in the solution for strains/stresses.

Energy Based Error Norms

The "Global Strain Energy Norm" $\|U\|^2$ and the "Global Energy Error Norm" $\|e\|^2$ can be expressed in terms of stresses using the following relations

$$\{\sigma\} = [E]\{\varepsilon\} \quad , \quad \{\sigma^*\} = [E]\{\varepsilon^*\}$$

Then

$$\|U\|^2 = \sum_{i=1}^{m} \{\sigma\}_i^T [E]^{-1} \{\sigma\}_i \, dv$$

$$\|e\|^2 = \sum_{i=1}^{m} \int \left(\{\sigma^*\}_i - \{\sigma\}_i\right)^T [E]^{-1} \left(\{\sigma^*\}_i - \{\sigma\}_i\right) dv$$

As alternative to $\|U\|^2$ and $\|e\|^2$ we can work with L_2-Norm of stresses alone and omit $[E]$ and $[E]^{-1}$

Then

$$\boxed{\begin{aligned} \|U\|_{L_2}^2 &= \sum_{i=1}^{m} \int \{\sigma\}_i^T \{\sigma\}_i \, dv \\ \|e\|_{L_2}^2 &= \sum_{i=1}^{m} \int \left(\{\sigma^*\}_i - \{\sigma\}_i\right)^T \left(\{\sigma^*\}_i - \{\sigma\}_i\right) dv \end{aligned}} \quad — (3)$$

The Relative Error can be defined as:

$$\boxed{\eta = \left[\frac{\|e\|^2}{\|U\|^2 + \|e\|^2} \right]^{1/2}} \quad —— (4)$$

The denominator in (4) is the estimate of Exact Energy

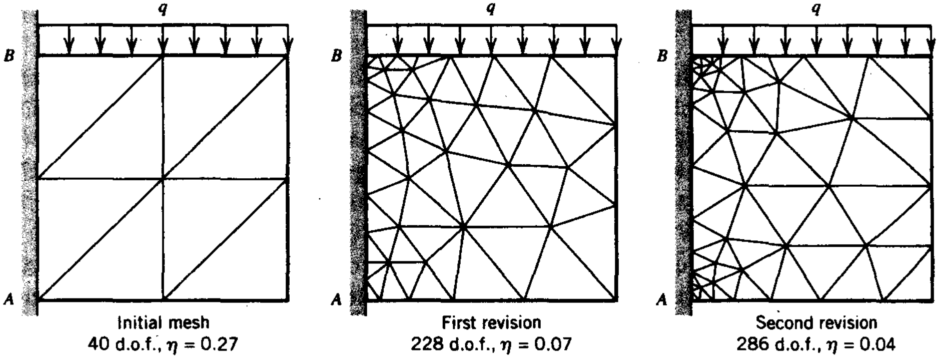


Figure 9.11-1. Results of an adaptive solution in a plane region using linear-strain triangles. Poisson's ratio is 0.3. All d.o.f. along *AB* are set to zero. Each mesh revision aimed at $\eta = 0.05$. [From J. Z. Zhu and O. C. Zienkiewicz, "Adaptive Techniques in the Finite Element Method," *Communications in Applied Numerical Methods*, Vol. 4, No. 2, 1988, pp. 197–204. © John Wiley & Sons Ltd. Reproduced by permission.]