# Matrix Analysis - Stiffness Method

## Force, Displacement and Stiffness Transformations from Energy Considerations

If $\{\Delta'\}$ = Vector of Nodal displacements in local or prime Coordinate system

$\{F'\}$ = Vector of Nodal Forces in local/prime Coords

$\{\Delta\}$ = " " " Displs. " Global Coords

$\{F\}$ = " " " Forces " " "

We have demonstrated that following Transformation relation holds between displacements in Local/prime Coords and Global Coordinates:

$$\{\Delta'\} = [T]\{\Delta\} \qquad (1)$$

Recall that we aim to transform the following Stiffness equations from Local/Primed Coordinates to Global Coordinates

$$\{F'\} = [K']\{\Delta'\} \qquad (2)$$

If the Local/Primed Coordinates and the Global Coordinates are both Orthogonal Coordinate systems ie each axis is ortogonal to the other axis in the coordinate system, then:

$$\text{Work done in the Local Coordinate System} = \frac{1}{2} F_i' \Delta_i' \qquad (3)$$

$$\text{Work done in the Global Coordinate System} = \frac{1}{2} F_i \Delta_i \qquad (4)$$

The work done in the local/primed coordinates and the Global Coordinates should be the same from consideration of "Energy Equivalence" and "Conservation of Energy". In Matrix notation we have:

$$\tfrac{1}{2}\{F'\}^T\{\Delta'\} = \tfrac{1}{2}\{F\}^T\{\Delta\} \quad \text{—— (5)}$$

$$\{F'\}^T\{\Delta'\} = \{F\}^T\{\Delta\} \quad \text{—— (6)}$$

Now we know that

$$\{\Delta'\} = [T]\{\Delta\} \quad \text{—— (7)}$$

Substitute Eqn (7) into Equation (6)

$$\{F'\}^T[T]\{\Delta\} = \{F\}^T\{\Delta\} \quad \text{—— (8)}$$

It follows that

$$\{F'\}^T[T] = \{F\}^T$$

$$\Rightarrow \quad \{F^T\}^T = \{F\} = \left\{\{F'\}^T[T]\right\}^T = [T]^T\{F'\}$$

$$\Rightarrow \quad \boxed{\{F\} = [T]^T\{F'\}} \quad \text{—— (9)}$$

Note that if we start from relation $\{\Delta'\} = [T]\{\Delta\}$ then Energy Equivalence requires that following relation between Forces exists $\{F\} = [T]^T\{F'\}$

ie

$$\boxed{\begin{array}{l}\text{If } \{\Delta'\} = [T]\{\Delta\} \\ \text{Then } \{F\} = [T]^T\{F'\}\end{array}} \quad \text{—— (10)}$$

Above Relation/Requirement is termed "Contragradience" and the transformations are termed "Contragradient" and are based on Principle of "Energy Equivalence"

## Stiffness Transformations from Energy Considerations

The work done in Local/primed coords and Global Coordinates is

$$W = \frac{1}{2}\{\Delta'\}^T\{F'\} = \frac{1}{2}\{\Delta\}^T\{F\}$$

Recall $\{\Delta'\} = [T]\{\Delta\} \Rightarrow \{\Delta'\}^T = \{\Delta\}^T[T]^T$

$$W = \frac{1}{2}\{\Delta\}^T[T]^T\{F'\} = \frac{1}{2}\{\Delta\}^T\{F\}$$

Also,

$$W = \frac{1}{2}\{\Delta'\}^T[K']\{\Delta'\} = \frac{1}{2}\{\Delta\}^T[K]\{\Delta\}$$

$$= \frac{1}{2}\{\Delta\}^T[T]^T[K'][T]\{\Delta\}$$

$$\Rightarrow \boxed{[K] = [T]^T[K'][T]}$$

Trasformed Stiffness Matrix

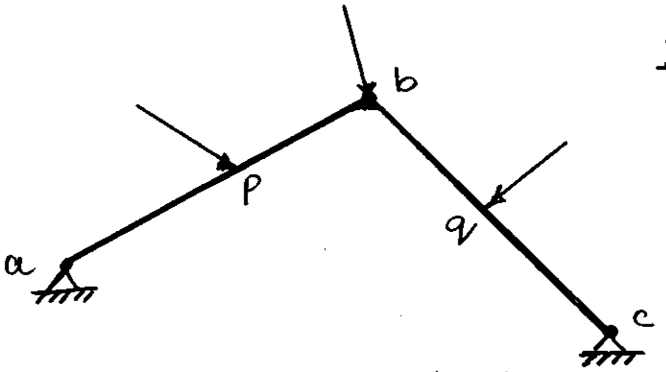
$$\boxed{\{F\} = [T]^T\{F'\}} \quad (11)$$

Force & Stiffness Matrix Transformation Relations from Energy Considerations.

## Case of Loads Applied between Member Nodes

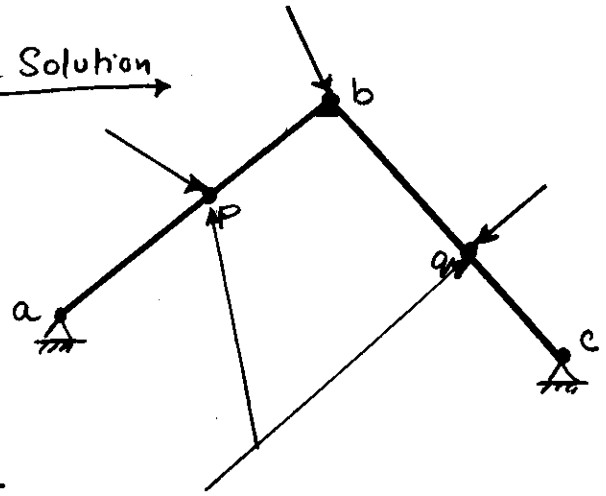
Till now we have looked at matrix analysis of structures in which the loads were applied at structure nodes.

Question arises as to how to carry out analysis in case of structures where loads are distributed over the lengths of members in a structure?? such as in the cases shown below:

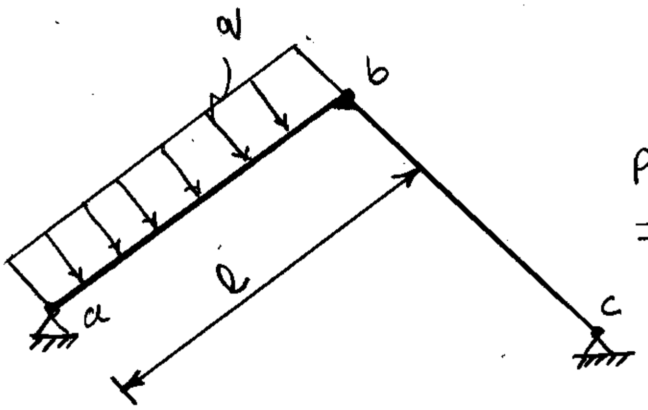



Concentrated Loads applied between member nodes

Possible Solution $\Longrightarrow$

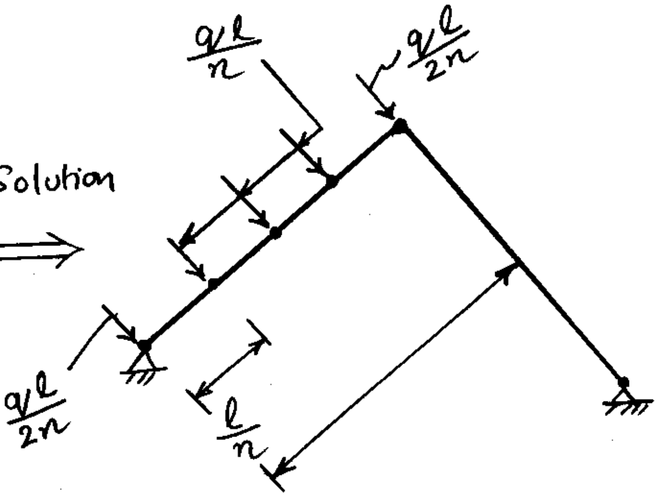



Introduction of nodes at Load Location




Distributed Load between member nodes

Possible Solution $\Longrightarrow$



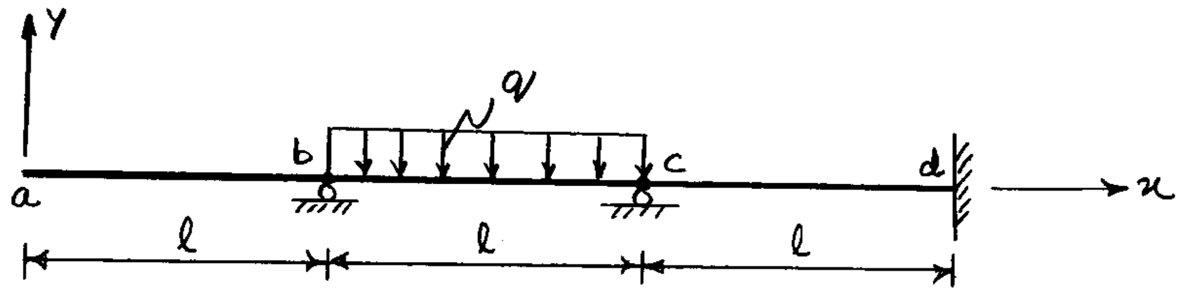

Introduction of addl nodes and replacement of distributed load with point loads

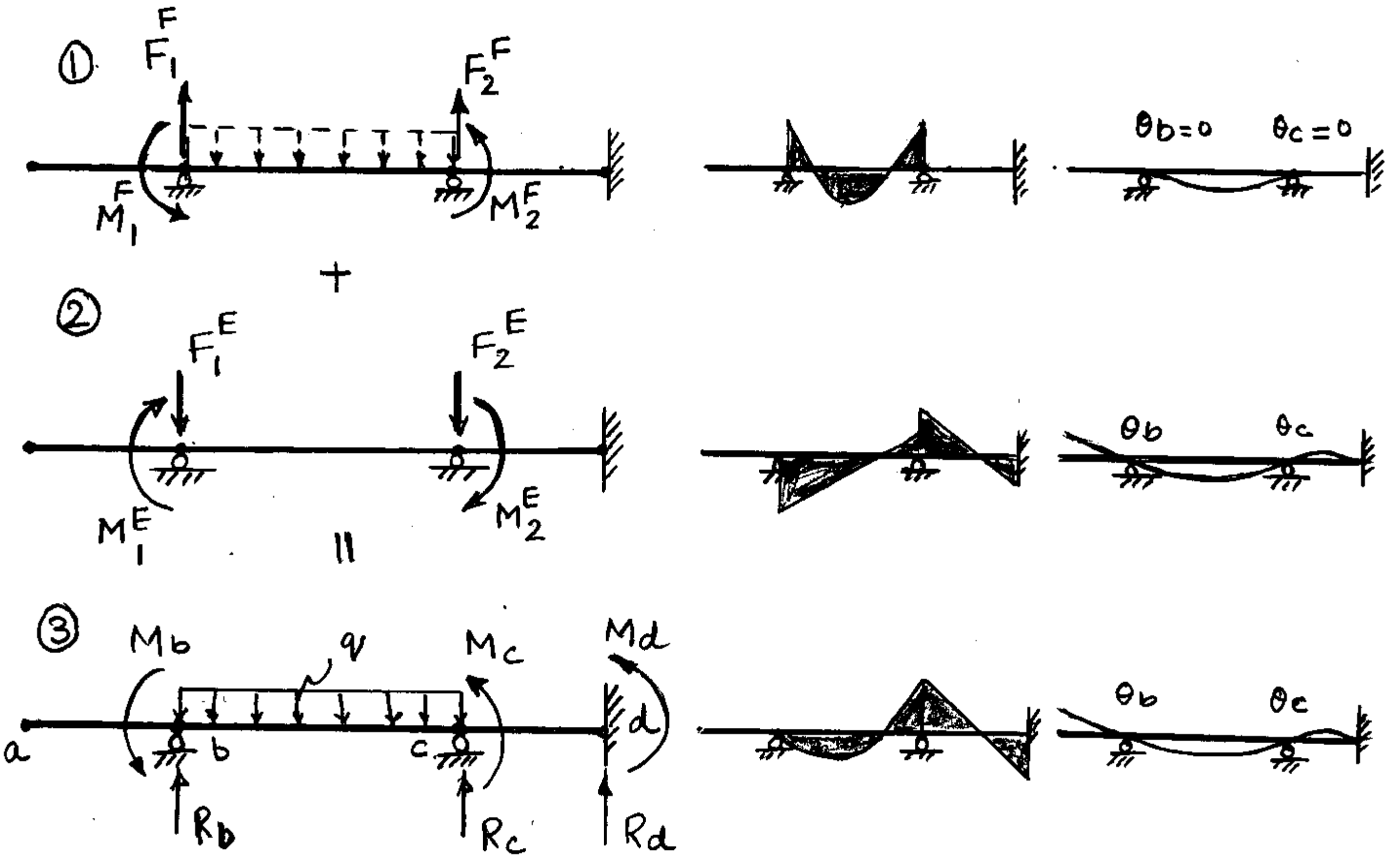
## Case of Loads Applied between Member Nodes

- The procedures/methods described previously would work, but at the cost of much extra computational effort as addinal degrees of freedoms have been introduced.
- An elegant and useful way of handling the case of loading applied between member nodes is available, which circumvents the need for introducing additional nodes and degrees of freedoms.
- This procedure is based on the concept of "Equivalent Nodal Loads". The Equivalent Nodal Loads being Nodal Loads that are applied to the existing member nodes, however, the equivalent nodal loads are computed in such a manner that the effect of distributed loads is transferred to the member end nodes.
- This procedure for generation of Equivalent Nodal Loads and for carrying out Structural Analysis is explained next by an example that illustrates the basic principles of the method of generation of "Equivalent Nodal Loads"

# Matrix Analysis – Stiffness Method

## Case of Loads Applied Between Member Nodes




① State in which Member Fixed End Forces & Moments are imposed on member ends

② State in which Equivalent Member Nodal Forces & Moments are imposed on structure nodes

③ Final/Solution member end Forces and Moments

| | | | | |
|---|---|---|---|---|
| Final Member End Forces | = | Member end Forces due to Equivalent Nodal Loads | + | Member Fixed End Forces |
| ③ | = | ② | + | ① |

# Matrix Analysis – Stiffness Method



## Case of Loads Applied between Member Nodes

- First we Lock the structure nodal degrees of freedom as a result of which Fixed End Forces develop at the member ends – This corresponds to state/Stage ① shown previously

- Next the Fictitious Constraints of locked nodes introduced in state/Stage ① are rectified by applying Equivalent Nodal Loads on the structure and carrying out a structural solution to determine nodal displacements and rotations. – This corresponds to state/stage ② shown previously.

  Note:
  - "Equivalent Member End Forces" are "Opposite" of Member Fixed End Forces
  - "The Equivalent Nodal Forces for the structure" are determined from the "Equivalent Member End Forces" by assembling them in the "Structural Nodal Load Vector" utilizing the "Member Destination Arrays"

- Member End Forces are recovered by superposing the member end forces determined in State ① and state ②

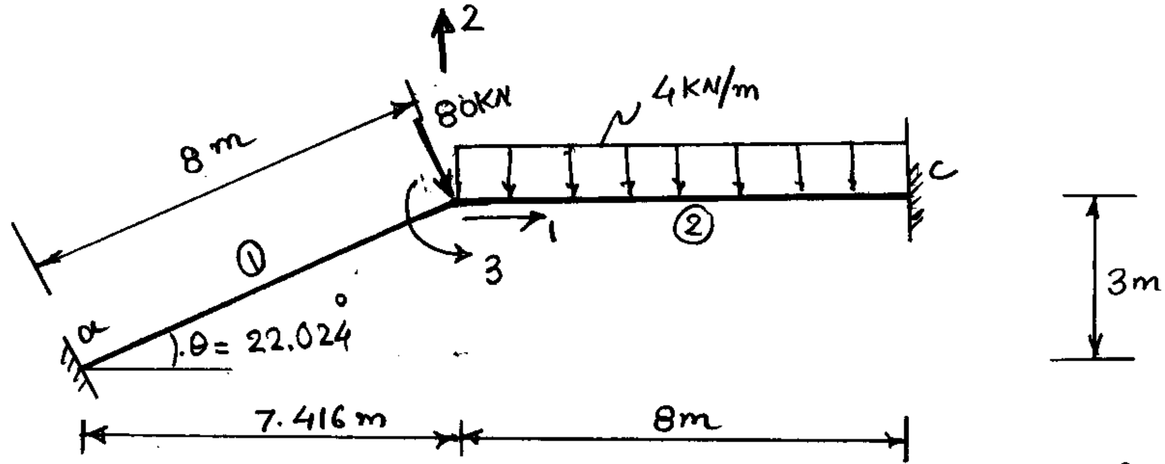
$$\{F\}^{Member}_{Final} = \{F\}^{Member}_{Eqiv\ Nodal\ Loads} + \{F\}^{Member}_{Fixed\ End\ Forces}$$

Compactly

$$\{F\}^{Member} = [K]_{Member}\,\{\Delta\}_{Member} + \{F_{FEM}\}_{Member}$$

## Example Problem

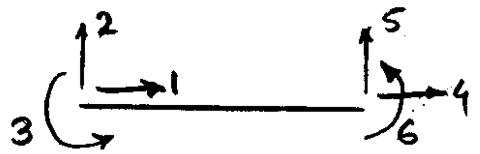
Analyze the frame shown below and determine member end forces and structural displacements. Take Axial deformations into account.




$A = 6000\ mm^2 = 0.006\ m^2$

$I = 200 \times 10^6\ mm^4 = 20 \times 10^{-5}\ m^4$

$E = 200{,}000\ N/mm^2 = 200\ KN/mm^2$




### Stiffness Matrix for a 4 DOF Member

$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ F_4 \\ F_5 \\ M_6 \end{Bmatrix} = E \begin{bmatrix} \frac{A}{L} & 0 & 0 & -\frac{A}{L} & 0 & 0 \\ 0 & \frac{12I}{L^3} & \frac{6I}{L^2} & 0 & -\frac{12I}{L^3} & \frac{6I}{L^2} \\ 0 & \frac{6I}{L^2} & \frac{4I}{L} & 0 & -\frac{6I}{L^2} & \frac{2}{L} \\ -\frac{A}{L} & 0 & 0 & \frac{A}{L} & 0 & 0 \\ 0 & -\frac{12}{L^3} & -\frac{6I}{L^2} & 0 & \frac{12I}{L^3} & -\frac{6I}{L^2} \\ 0 & \frac{6}{L^2} & \frac{2I}{L} & 0 & -\frac{6I}{L^2} & \frac{4I}{L} \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \theta_3 \\ \Delta_4 \\ \Delta_5 \\ \theta_6 \end{Bmatrix}$$

## Example Problem

**Member ① - ab**

Stiffness Matrix

$$\frac{A}{L} = \frac{6000}{8000} = 0.75 \text{ mm}$$

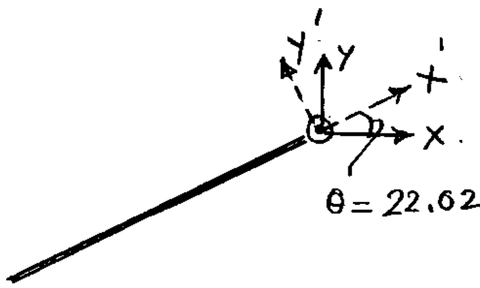
$$\frac{12I}{L^3} = \frac{12 \times 200 \times 10^6}{(8000)^3} = 0.00469 \text{ mm}$$

$$\frac{6I}{L^2} = \frac{6 \times 200 \times 10^6}{(8000)^2} = 18.75 \text{ mm}$$

$$\frac{4I}{L} = \frac{4 \times 200 \times 10^6}{8000} = 100,000 \text{ mm}$$

$$\frac{2I}{L} = \; = 50,000 \text{ mm}$$

$$[K_{①}]_{Local} = 200 \begin{bmatrix} 0.75 & 0 & 0 & -0.75 & 0 & 0 \\ 0 & 0.00469 & 18.75 & 0 & -0.00469 & 18.75 \\ 0 & 18.75 & 100,000 & 0 & -18.75 & 50,000 \\ -0.75 & 0 & 0 & 0.75 & 0 & 0 \\ 0 & -0.00469 & -18.75 & 0 & 0.00469 & -18.75 \\ 0 & 18.75 & 50,000 & 0 & -18.75 & 100,000 \end{bmatrix}$$




$\theta = 22.024°$

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x'$ | $\cos\theta$ | $\sin\theta$ | 0 |
| $y'$ | $-\sin\theta$ | $\cos\theta$ | 0 |
| $z'$ | 0 | 0 | 1 |

$$[\gamma] = \begin{bmatrix} \cos 22.024 & \sin 22.024 & 0 \\ -\sin 22.024 & \cos 22.024 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

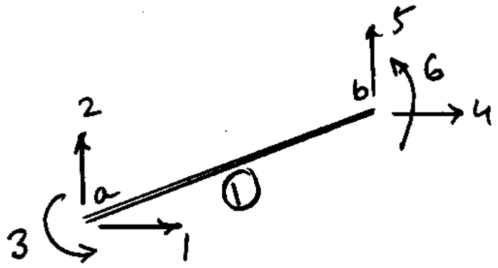
$$[\gamma]_{3\times3} = \begin{bmatrix} 0.927 & 0.375 & 0 \\ -0.375 & 0.927 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# Matrix Analysis – Stiffness Method

## Member ① – ab

$$\text{Transformation Matrix} = [T]_{6\times6} = \left[\begin{array}{c|c} [\gamma]_{3\times3} & 0 \\ \hline 0 & [\gamma]_{3\times3} \end{array}\right]$$

$$[T]_{6\times6} = \left[\begin{array}{ccc|ccc} 0.927 & 0.375 & 0 & & & \\ -0.375 & 0.927 & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hline & & & 0.927 & 0.375 & 0 \\ & 0 & & -0.375 & 0.927 & 0 \\ & & & 0 & 0 & 1 \end{array}\right]_{6\times6}$$



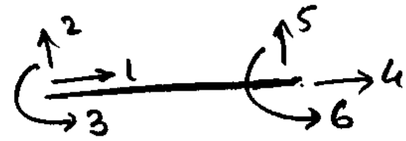

$$[K①]_{Global} = [T]^T \, [K'①]_{Local} \, [T]$$

$$= 200 \begin{array}{c} \begin{array}{cccccc} 0 & 0 & 0 & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccccc} 0.6452 & 0.2591 & -7.0313 & -0.6452 & -0.2591 & -7.0313 \\ & 0.1095 & 17.381 & -0.2591 & -0.1095 & 17.381 \\ & & 100{,}000 & 7.0313 & -17.381 & 50{,}000 \\ & & & 0.6452 & 0.2591 & 7.0313 \\ & \text{sym} & & & 0.1095 & -17.381 \\ & & & & & 100{,}000 \end{array}\right] \end{array} \begin{array}{c} \\ 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ 3 \end{array}$$

$$DA① = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ 3 \end{Bmatrix}$$

## Example Problem

**Member ② - bc**




The Stiffness matrix for this element is same as member ① local stiffness matrix

$$[K②]_{local} = [K②]_{Global}$$

$$= \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 0 & 0 & 0 \end{array} \\ \begin{bmatrix} 0.75 & 0 & 0 & -0.75 & 0 & 0 \\ & 0.00469 & 18.75 & 0 & -0.00469 & 18.75 \\ & & 100{,}000 & 0 & -18.75 & 50{,}000 \\ & & & 0.75 & 0 & 0 \\ & \text{Sym} & & & 0.00469 & -18.75 \\ & & & & & 100{,}000 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{array}$$

$$DA② = \begin{Bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{F_{FEM②}\} = \begin{Bmatrix} 0 \\ \frac{wL}{2} \\ \frac{wl^2}{12} \\ 0 \\ \frac{wL}{2} \\ -\frac{wl^2}{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{4}{1000} \times \frac{8000}{2} \\ \frac{\frac{4}{1000} \times (8000)^2}{12} \\ 0 \\ ------ \\ ------ \end{Bmatrix} = \begin{Bmatrix} 0 \\ 16 \\ 21330 \\ 0 \\ 16 \\ -21330 \end{Bmatrix} \text{kN-mm}$$

$$\{F_{EQV②}\} = \begin{Bmatrix} 0 \\ -16 \\ -21330 \\ 0 \\ -16 \\ 21330 \end{Bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 0 \\ 0 \\ 0 \end{array}$$

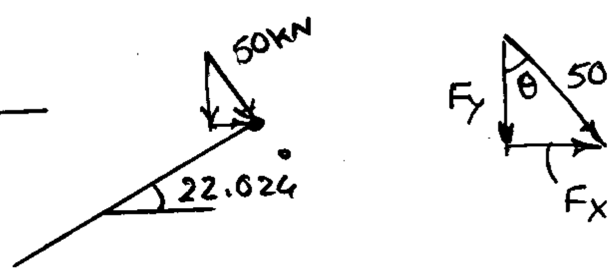
## Example Problem

### Global Structure Stiffness Matrix

$$[K_G]_{\text{Structure}} = 200 \begin{array}{ccc} 1 & 2 & 3 \\ \left[\begin{array}{ccc} 0.645 + 0.75 & 0.2591 & 7.0313 \\ & 0.1095 + 0.00469 & -17.381 + 18.75 \\ \text{Sym} & & 100{,}000 + 100{,}000 \end{array}\right] \end{array}$$

$$[K_G]_{\text{Structure}} = 200 \begin{array}{c} \left[\begin{array}{ccc} 1.395 & 0.2591 & 7.0313 \\ 0.2591 & 0.1142 & 1.369 \\ 7.0313 & 1.369 & 200{,}000 \end{array}\right] \end{array} \begin{array}{c} \Delta_1 \\ \Delta_2 \\ \theta_3 \end{array}$$

### Resolve Nodal force @ pt b in Global Coords




$$F_x = 50 \sin 22.024 = 18.75 \rightarrow$$

$$F_y = -50 \cos 22.024 = -46.35 \downarrow$$

### Structural Nodal Load Vector

$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \end{Bmatrix} = \begin{Bmatrix} 18.75 + 0 \\ -46.35 - 16.0 \\ 0 - 21330 \end{Bmatrix} = \begin{Bmatrix} 18.75 \\ -62.35 \\ -21330 \end{Bmatrix} \begin{array}{l} \text{KN} \\ \text{KN} \\ \text{KN-mm} \end{array}$$

## Example Problem

We can now write the structure equilibrium Equation.

$$[K]\{\Delta\} = \{P\}$$

$$200\begin{bmatrix} 1.395 & 0.2591 & 7.0313 \\ & 0.1142 & 1.369 \\ \text{Sym} & & 200{,}000 \end{bmatrix}\begin{Bmatrix}\Delta_1\\ \Delta_2 \\ \theta_3\end{Bmatrix} = \begin{Bmatrix}18.75\\ -62.35\\ -21330\end{Bmatrix}$$

$$\{\Delta\} = [K]^{-1}\{P\}$$

$$\Rightarrow \begin{Bmatrix}\Delta_1\\ \Delta_2\\ \theta_3\end{Bmatrix} = \begin{Bmatrix}0.9982\ \text{mm}\\ -4.996\ \text{mm}\\ -0.000534\ \text{rad}\end{Bmatrix}$$

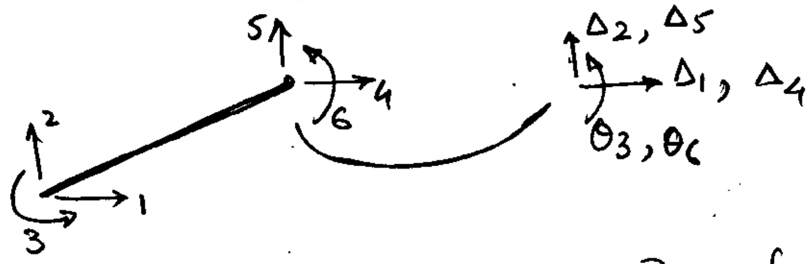
## End Reactions

Use Member Stiffness matrices to determine end reactions

Recall

$$\{F\}_{\text{Member}} = [K]_{\text{member}}\{\Delta\}_{\text{member}} + \{F_{FEM}\}_{\text{Member}}.$$

Member ①




$$200\begin{bmatrix} -0.6452 & -0.2591 & -7.0313 \\ -0.2591 & -0.1095 & 17.381 \\ 7.0313 & -17.381 & 50{,}000 \end{bmatrix}\begin{Bmatrix}0.9982\\ -4.996\\ -0.000534\end{Bmatrix} + \begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

$$\begin{Bmatrix}R_{ax}\\ R_{ay}\\ M_{az}\end{Bmatrix} = \begin{Bmatrix}131.0\\ 55.4\\ 13.43\times10^{3}\end{Bmatrix}\begin{matrix}\text{KN}\\ \text{KN}\\ \text{KN-mm}\end{matrix}$$

# Matrix Analysis - Stiffness Method

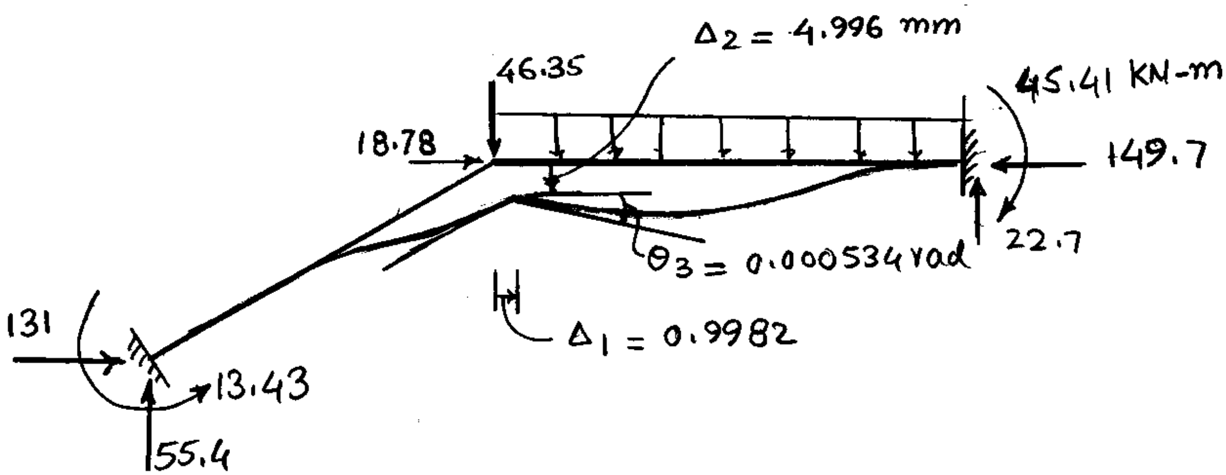
## Example Problem

### Member (2) Reactions

$$200\begin{bmatrix} -0.75 & 0 & 0 \\ 0 & -0.00469 & -18.75 \\ 0 & 18.75 & 50{,}000 \end{bmatrix}\begin{Bmatrix} 0.9982 \\ -4.996 \\ -0.000534 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 16 \\ -21330 \end{Bmatrix}$$

$[K]$ $\{\Delta\}$ $F_{FEM}$

$$\begin{Bmatrix} R_{cx} \\ R_{cy} \\ M_{cz} \end{Bmatrix} = \begin{Bmatrix} -149.7 \\ 22.7 \\ -45.41\times10^3 \end{Bmatrix} \begin{matrix} \text{KN} \\ \text{KN} \\ \text{KN-mm} \end{matrix}$$




Deformed Shape & Member end Reactions