Static Condensation

- "Static Condensation" or Condensation refers to reduction/contraction in the number of degrees of freedom by mathematical operations on the assembled Structure Equilibrium Equations.

- In the Static Condensation Process a reduced set of Structure Equilibrium Equations are arrived at by eliminating a selected set of degrees of freedoms $\{\Delta_b\}$ and retaining only the desired degrees of freedom $\{\Delta_c\}$

- The Mathematical Description of Static Condensation Process is as follows:

The Structure Equilibrium Equations are of the form:

$$[K]\{\Delta\} = \{P\} \quad \text{——— (1)}$$

which in Partitioned form may be written as:

$$\left[\begin{array}{c|c} K_{bb} & K_{bc} \\ \hline K_{cb} & K_{cc} \end{array}\right] \left\{\begin{array}{c} \Delta_b \\ \hline \Delta_c \end{array}\right\} = \left\{\begin{array}{c} P_b \\ \hline P_c \end{array}\right\} \quad \text{——— (2)}$$

We aim to eliminate degrees of freedom $\{\Delta_b\}$

- Expanding the upper partition of the Equations (2) we have:

$$[K_{bb}]\{\Delta_b\} + [K_{bc}]\{\Delta_c\} = \{P_b\}$$

$$\Rightarrow \boxed{\begin{array}{l} \{\Delta_b\} = [K_{bb}]^{-1}\left[\{P_b\} - [K_{bc}]\{\Delta_c\}\right] \\ \text{or} \quad \Delta_b = K_{bb}^{-1}\left(P_b - K_{bc}\Delta_c\right) \end{array}} \quad \text{——— (3)}$$

Static Condensation

Expanding the lower portion of Stiffness Equations ②

$$K_{cb}\,\Delta_b + K_{cc}\,\Delta_c = P_c \quad \text{— ④}$$

Substituting Δ_b from Eqn ③ into Eqn ④ above we have

$$K_{cb}\left[K_{bb}^{-1}\left[P_b - K_{bc}\Delta_c\right]\right] + K_{cc}\,\Delta_c = P_c$$

$$-K_{cb}\,K_{bb}^{-1}\,K_{bc}\,\Delta_c + K_{cc}\,\Delta_c = P_c - K_{cb}\,K_{bb}^{-1}\,P_b$$

$$\underbrace{\left[-[K_{cb}][K_{bb}]^{-1}[K_{bc}] + K_{cc}\right]}_{[\hat{K}_{cc}]}\{\Delta_c\} = \underbrace{\{P_c\} - [K_{cb}][K_{bb}]^{-1}\{P_b\}}_{\{\hat{P}_c\}}$$

With the above definitions we can write the following condensed set of equations

$$\boxed{[\hat{K}_{cc}]\{\Delta_c\} = \{\hat{P}_c\}} \quad \text{— ⑤}$$

Where

$$\hat{K}_{cc} = -K_{cb}\,K_{bb}\,K_{bc} + K_{cc}$$

$$\hat{P}_c = P_c - K_{cb}\,K_{bb}\,P_b$$

Condensed Set of Equations

Note: In the above condensed set of equations $\{\Delta_c\}$ have been retained and $\{\Delta_b\}$ DOFs have been been "eliminated/condensed out".

Static Condensation

$$[\hat{K_{cc}}]\{\Delta_c\} = \{\hat{P_c}\}$$

Once the condensed degrees of freedom $\{\Delta c\}$ are determined from the above condensed set of Equations by inversion or some other process as follows:

$$\{\Delta c\} = [\hat{K_{cc}}]^{-1}\{\hat{P_c}\} \quad \text{——— ⑥}$$

The eliminated degrees of freedoms can be determined by sobstituting $\{\Delta c\}$ from ⑥ into Equation ③

$$\Delta_c = K_{bb}^{-1}(P_b - K_{bc}\,\Delta_c) \quad \text{——— ③}$$

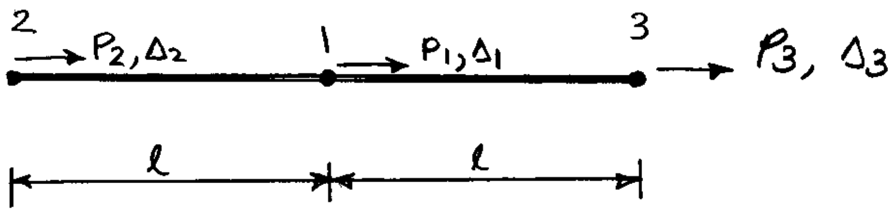
$$\Rightarrow \boxed{\begin{aligned} \{\Delta c\} &= [K_{bb}]^{-1}\left[\{P_b\} - [K_{bc}][\hat{K_{cc}}]^{-1}\{\hat{P_c}\}\right] \\ \Delta c &= K_{bb}^{-1}\left[P_b - K_{bc}\,\hat{K_{cc}}\,\hat{P_c}\right] \end{aligned}} \quad \text{——— ⑥}$$

Recovery of Eliminated DOFs

- Static Condensation may be employed if a Large structure having Large number of degrees of freedoms has to be solved.
- Static Condenstation reduces the overall size of the system of Structure Equilibrium Equations that has to be solved.
- The eliminated DOFs can be recovered later after solving for reduced/condensed degrees of freedoms.
- Although, the number of matrix manipulations increases the condensed system of equations can be easily handled by equation solvers as computer memory requirements are reduced.

Static Condensation

Example & Physical Interpretation.



Consider an assemblage of two truss elements shown above having 3 degrees of freedoms. And we wish to eleminate degree of freedom 3.

The element stiffness matrix for both the elements is

$$K_{①} = K_{②} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad , \quad DA① = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$$

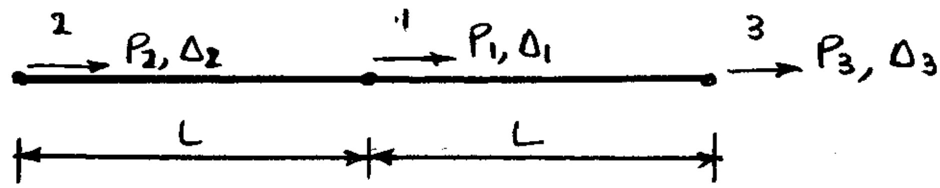
$$DA② = \begin{Bmatrix} 1 \\ 3 \end{Bmatrix}$$

The Structure Stiffness Matrix and Equilibrium Equations are:

$$\frac{AE}{L}\begin{bmatrix} 1+1 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\boxed{\frac{AE}{L}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}} \quad \text{——— ①}$$

Static Condensation



$$\frac{AE}{L}\begin{bmatrix}2 & -1 & -1\\ -1 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}\begin{Bmatrix}\Delta_1\\ \Delta_2\\ \Delta_3\end{Bmatrix} = \begin{Bmatrix}P_1\\ P_2\\ P_3\end{Bmatrix}$$

By Gauss Elimination we can condense out Degree of Freedom Δ_1

$$\begin{matrix}R_2 - \frac{R_1}{2}\\ R_3 - \frac{R_1}{2}\end{matrix} \Rightarrow \frac{AE}{L}\left[\begin{array}{c|cc}2 & -1 & -1\\ \hline 0 & 0.5 & -0.5\\ 0 & -0.5 & 0.5\end{array}\right]\begin{Bmatrix}\Delta_1\\ \hline \Delta_2\\ \Delta_3\end{Bmatrix} = \begin{Bmatrix}P_1\\ \hline P_2 - P_1/2\\ P_3 - P_1/2\end{Bmatrix}$$

From the Lower Partition of the Equilibrium Equations we have:

$$\boxed{\frac{AE}{2L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\Delta_2\\ \Delta_3\end{Bmatrix} = \begin{Bmatrix}P_2 - P_1/2\\ P_3 - P_1/2\end{Bmatrix}} \quad \text{— } (2)$$

Condensed Set of Eqns.

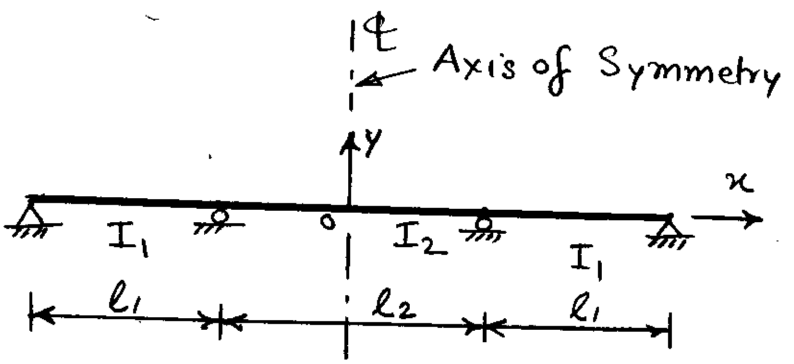
Now if $P_1 = 0$, the above Equations reduce to:

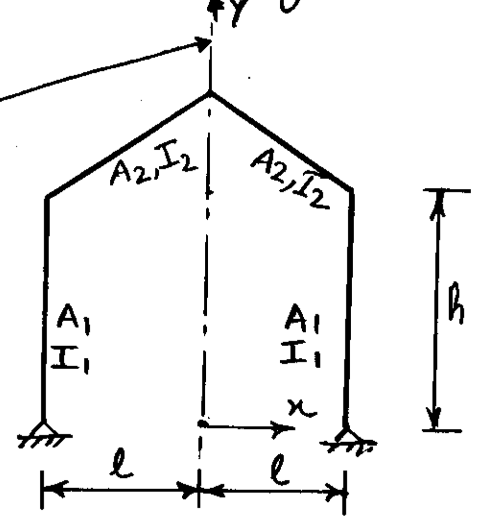
$$\boxed{\frac{AE}{2L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\Delta_2\\ \Delta_3\end{Bmatrix} = \begin{Bmatrix}P_2\\ P_3\end{Bmatrix}} \quad \text{—— } (3)$$

* * We immediately recognize that in the above Eqns (2) & (3) the stiffness matrix is the stiffness matrix for bar element with length equal to "2L".
* * Thus, the condensation process is equivalent to releasing the degrees of freedoms selected for elimination/condensing out.
* * The Loads if present at the eliminated DOFs are distributed to the Condensed DOFs as in Eqn (2)

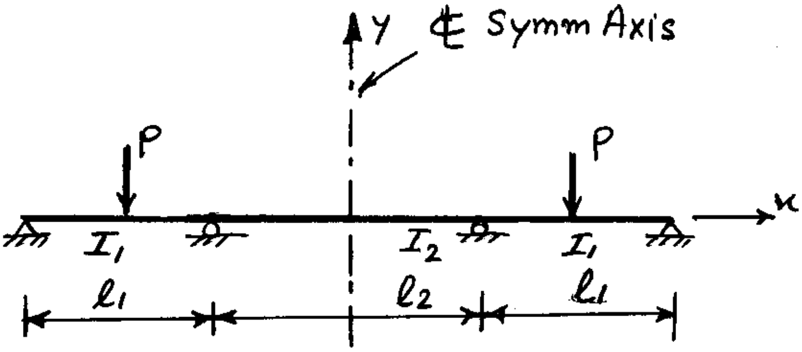
Symmetry & Antisymmetry

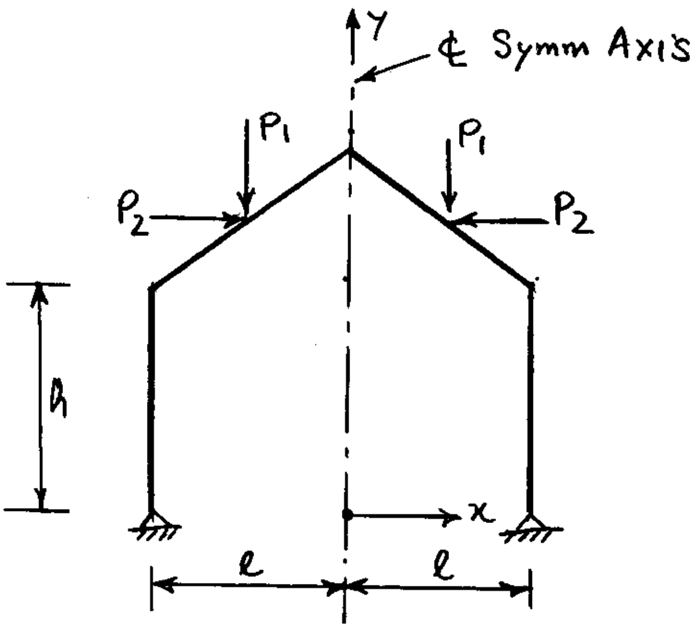
- Many structures such as buildings, bridges etc. possess some form of symmetry, which can be utilized to considerably reduce computational effort for analysis of the structure.
- Consideration of symmetry and antisymmetry allows us to analyze only a portion of the whole structure as the symmetric analyzed portion of the structure represents the entire structure.
- Utilization of symmetry/antisymmetry involves considering the following aspects:
  1. Recognition and definition of the type of symmetry
  2. Manipulation of loads and forces in such a way that advantage of symmetry/antisymmetry can be taken.
  3. Prescription of proper boundary conditions on the isolated symmetric portion of the whole structure



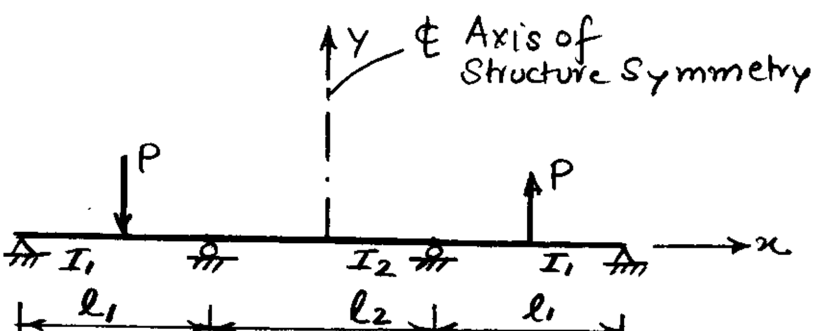


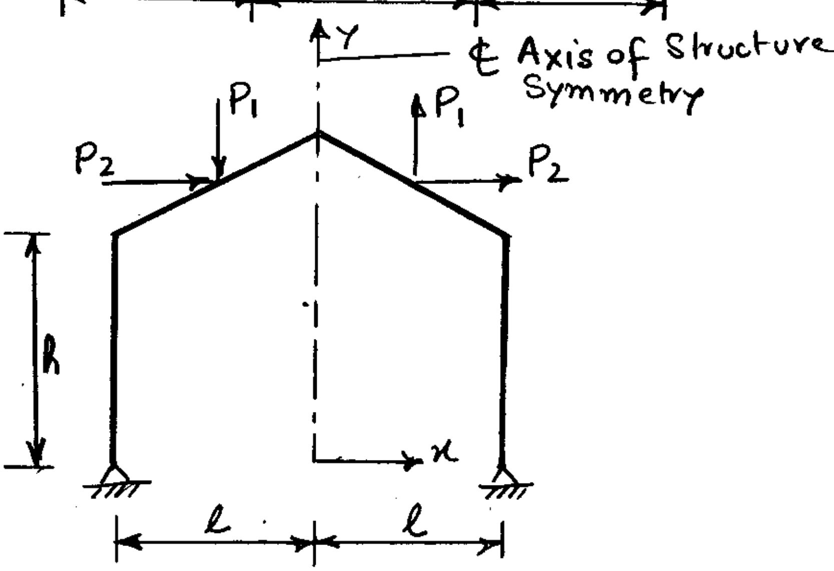
Examples of Structures that are structurally Symmetric





Examples of Symmetric Loadings





Examples of Antisymmetric Loadings

Symmetry & Antisymmetry

Boundary Conditions For Symmetry & Antisymmetry

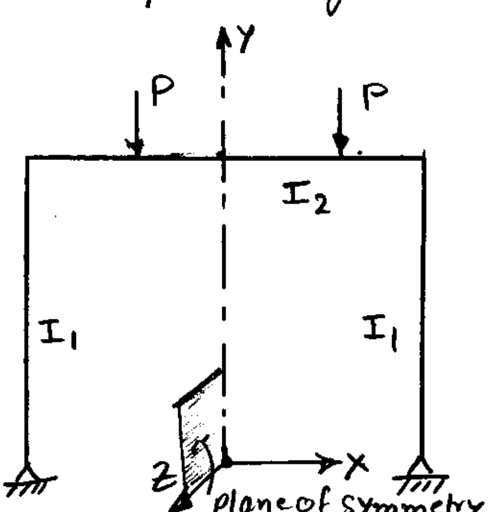
Taking advantage of symmetry or antisymmetry of loadings, one needs to model only part of the whole structure. The problem that arises then is "what boundary conditions to apply at the planes of structural symmetry for analysis of the structure??

In general, following displacement constraints need to be applied at planes of symmetry:

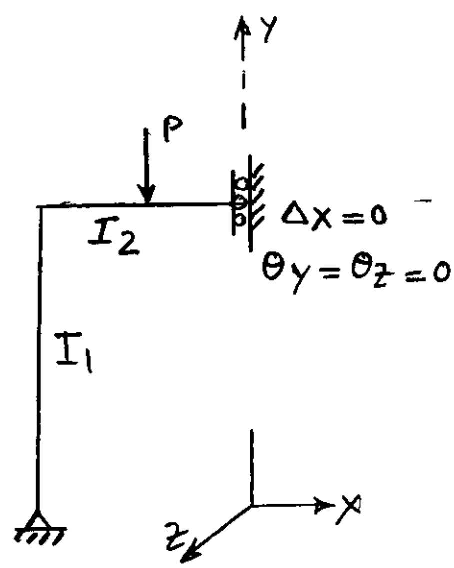
Symmetric Structures - Symmetric Loads

a) No translation normal to the plane of symmetry

b) No rotations about two orthogonal axes in the plane of symmetry.



Symmetric Structure
" Loading



Part Model of Symmetric Portion

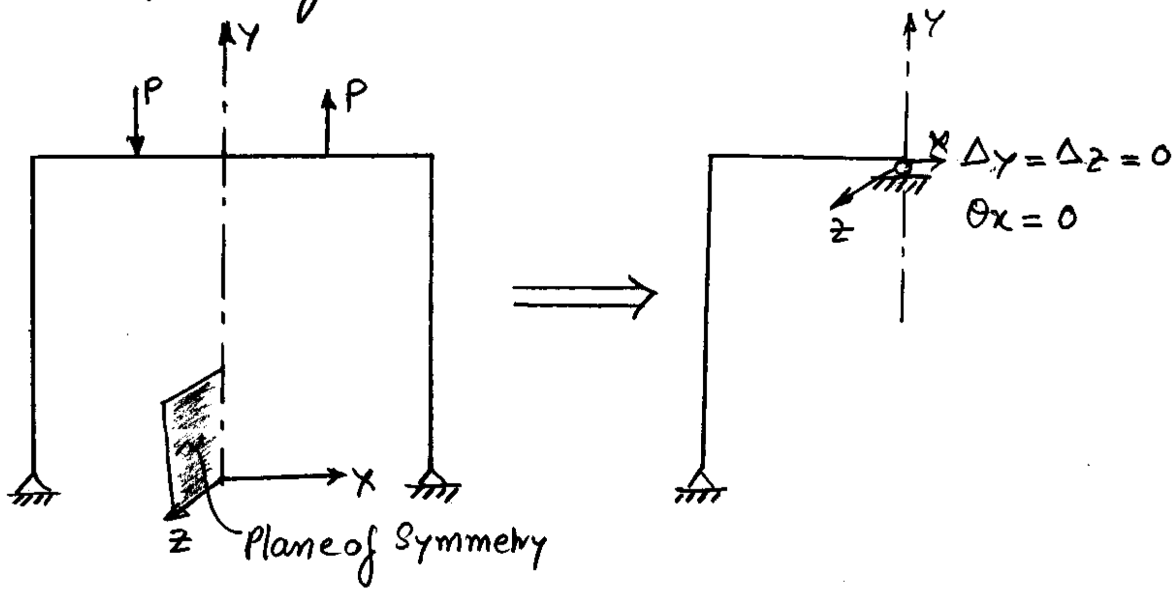
Matrix Analysis - Stiffness Method

Symmetric Structures - Antisymmetric Loads

Boundary Conditions @ planes of antisymmetry.

a) No translation in the plane of symmetry

b) No rotation about an axis normal to the plane of symmetry.



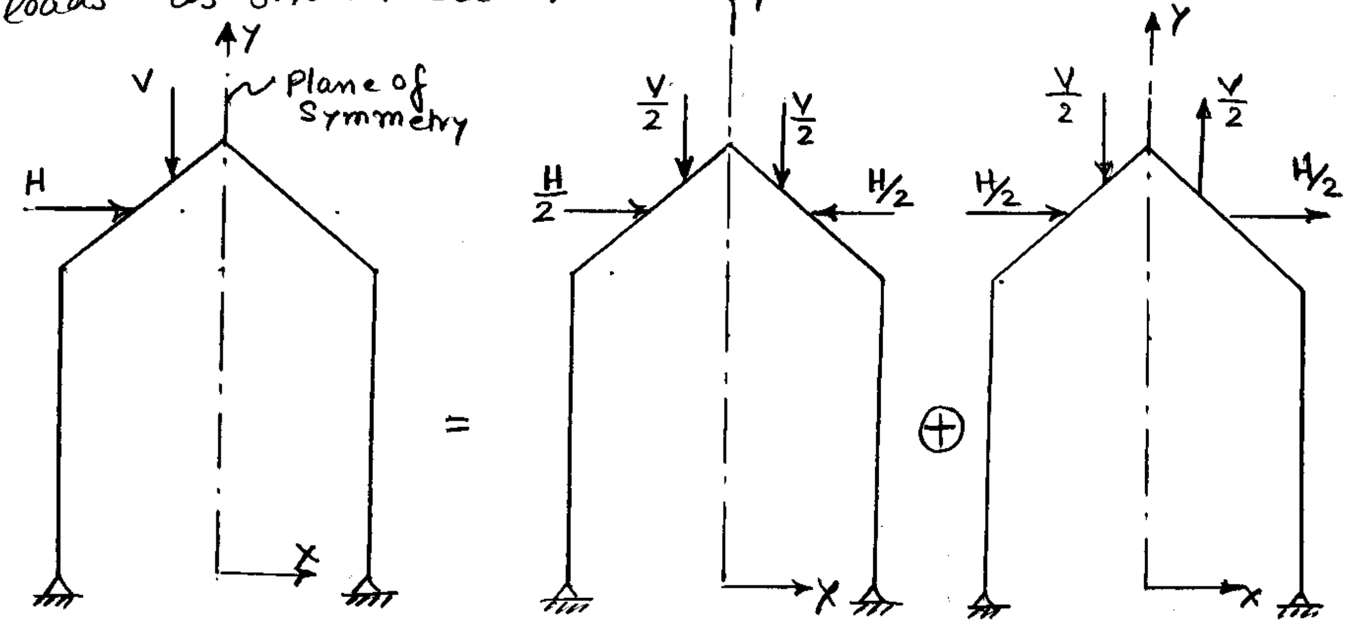
Symmetric Structure & Antisymmetric Loading

Part Model of Symmetric Portion.

Matrix Analysis- Stiffness Method

Symmetry & Antisymmetry

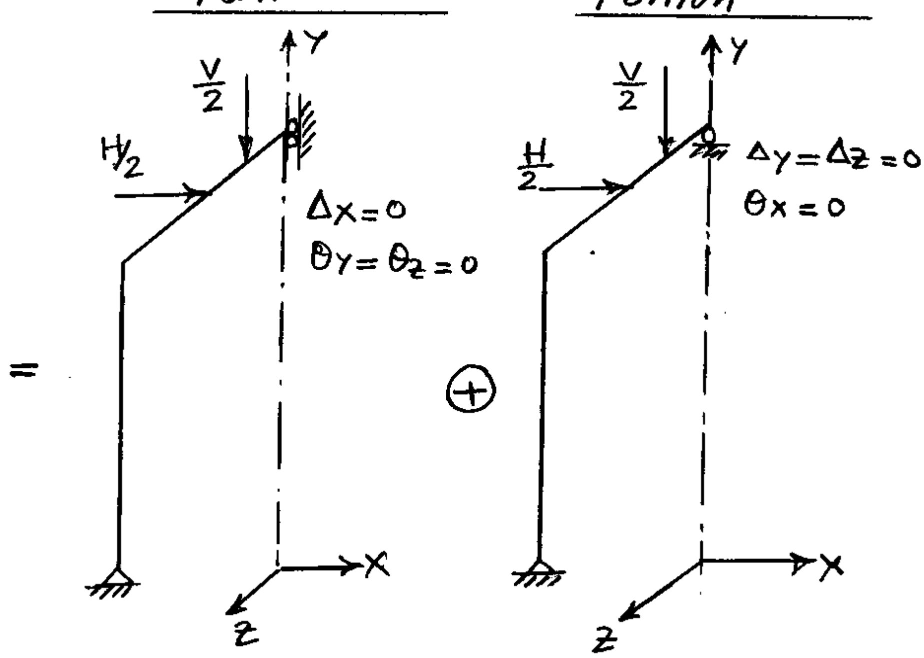
Many times, loads on a symmetric structure may neither be symmetric nor antisymmetric. However, it may still be possible to model only a portion of the structure and represent the applied loads as combination of symmetric and antisymmetric loads as shown below:



Actual Loading

Symmetric Portion

Antisymmetric Portion



Symmetric Model

Antisymmetric Model

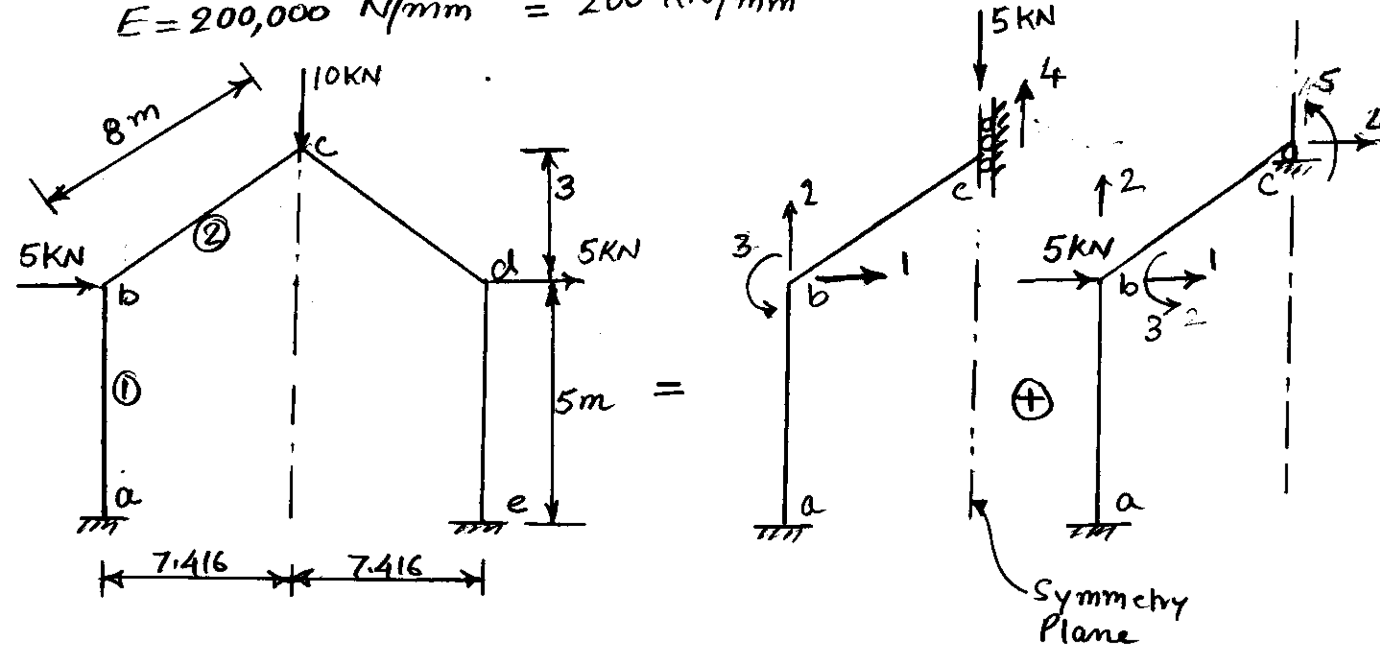
Example Problem

Using principles of symmetry and antisymmetry, determine the displacements at the joints of the rigid frame shown below. Neglect Axial Deformations.

Member ab & de $\quad A = 4 \times 10^3\ mm^2$, $I = 50 \times 10^6\ mm^4$

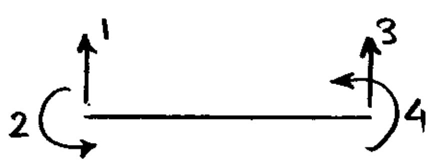
Member bc & cd $\quad A = 6 \times 10^3\ mm^2$, $I = 200 \times 10^6\ mm^4$

$E = 200{,}000\ N/mm^2 = 200\ KN/mm^2$



Full Structure & Loading

Symmetric Loading ⊕ Antisymmetric Loading

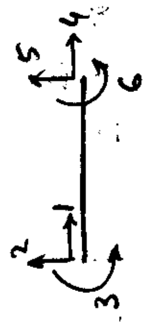


$$K_{\substack{Frame\\Typical}} = E \begin{bmatrix} \frac{A}{L} & 0 & 0 & -\frac{A}{L} & 0 & 0 \\ 0 & \frac{12I}{L^3} & \frac{6I}{L^2} & 0 & -\frac{12I}{L^3} & \frac{6I}{L^2} \\ 0 & \frac{6I}{L^2} & \frac{4I}{L} & 0 & -\frac{6I}{L^2} & \frac{2I}{L} \\ -\frac{A}{L} & 0 & 0 & \frac{A}{L} & 0 & 0 \\ 0 & -\frac{12I}{L^3} & -\frac{6I}{L^2} & 0 & \frac{12I}{L^3} & -\frac{6I}{L^2} \\ 0 & \frac{6I}{L^2} & \frac{2I}{L^2} & 0 & -\frac{6I}{L^2} & \frac{4I}{L} \end{bmatrix}$$

Matrix Analysis – Stiffness Method

Example Problem – Symmetry

Member ab



$$\frac{A}{L} = \frac{4 \times 10^3}{5000} = 0.8$$

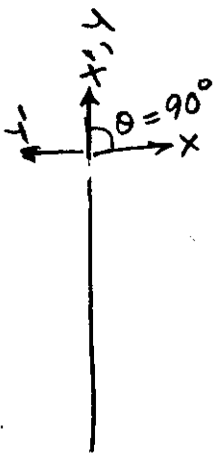
$$\frac{12I}{L^3} = \frac{12 \times 50 \times 10^6}{(5000)^3} = 0.0048 \quad \text{mm}$$

$$\frac{6I}{L^2} = \frac{6 \times 50 \times 10^6}{(5000)^2} = 12.0$$

$$\frac{4I}{L} = \frac{4 \times 50 \times 10^6}{5000} = 40{,}000$$

$$\frac{2I}{L} = 20{,}000$$

$$K_1^{ab} = 200 \begin{bmatrix} 0.8 & 0 & 0 & -0.8 & 0 & 0 \\ & 0.0048 & 12 & 0 & -0.0048 & 12 \\ & & 40{,}000 & 0 & -12 & 20{,}000 \\ & & & 0.8 & 0 & 0 \\ & & & & 0.0048 & -12 \\ & & & & & 40{,}000 \end{bmatrix}$$



	x	y	z
x'	0	1	0
y'	−1	0	0
z	0	0	1

Member ab

$$T = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & & & \\ -1 & 0 & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hline & & & 0 & 1 & 0 \\ & 0 & & -1 & 0 & 0 \\ & & & 0 & 0 & 1 \end{array}\right]$$

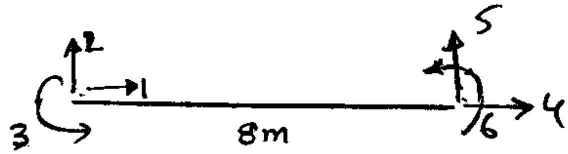
$$K_{Gab} = T^T K'_{ab} T$$

$$= 200 \begin{array}{c} \begin{array}{cccccc} & & & 1 & 2 & 3 \end{array} \\ \left[\begin{array}{cccccc} 0.0048 & 0 & -12 & -0.0048 & 0 & -12 \\ 0 & 0.80 & 0 & 0 & -0.8 & 0 \\ -12 & 0 & 40{,}000 & 12 & 0 & 20{,}000 \\ -0.0048 & 0 & 12 & 0.0048 & 0 & 12 \\ 0 & -0.8 & 0 & 0 & 0.8 & 0 \\ -12 & 0 & 20{,}000 & 12 & 0 & 40{,}000 \end{array}\right] \end{array} \begin{array}{c} \\ \\ \\ \\ 1 \\ 2 \\ 3 \end{array}$$

$$DA = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ 3 \end{Bmatrix}$$

Example Problem – Symmetry

Member bc



$$\frac{12I}{L^3} = \frac{12 \times 200 \times 10^6}{(8000)^3} = 0.00469 \qquad \frac{A}{L} = \frac{6 \times 10^3}{8000} = 0.75$$

$$\frac{6I}{L^2} = \frac{6 \times 200 \times 10^6}{(8000)^2} = 18.75$$

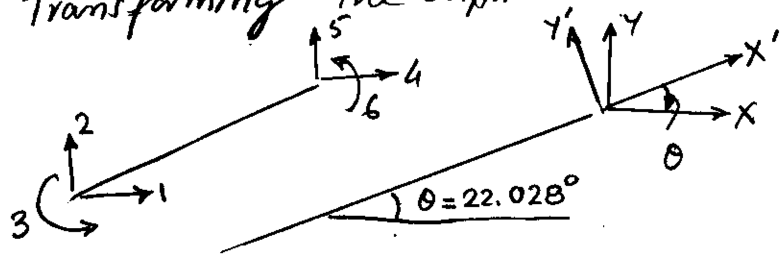
$$\frac{4I}{L} = \frac{4 \times 200 \times 10^6}{8000} = 100{,}000$$

$$\frac{2I}{L} = \quad = 50{,}000$$

$$K_{bc} \; ② = 200 \begin{bmatrix} 0.75 & 0 & 0 & -0.75 & 0 & 0 \\ 0 & 0.00469 & 18.75 & 0 & -0.00469 & 18.75 \\ 0 & 18.75 & 100{,}000 & 0 & -18.75 & 50{,}000 \\ -0.75 & 0 & 0 & 0.75 & 0 & 0 \\ 0 & -0.00469 & -18.75 & 0 & 0.00469 & -18.75 \\ 0 & 18.75 & 50{,}000 & 0 & -18.75 & 100{,}000 \end{bmatrix}$$

Member bc

Transforming the Stiffness Matrix to Global Coordinates



	x	Y	z
x'	$l_1 = \cos\theta$	$m_1 = \sin\theta$	0
y'	$l_2 = -\sin\theta$	$m_2 = \cos\theta$	0
z'	$l_3 = 0$	$m_3 = 0$	1

$$T = \left[\begin{array}{c|c} [\gamma] & 0 \\ \hline 0 & [\gamma] \end{array}\right]$$

$$[T] = \left[\begin{array}{ccc|ccc} 0.927 & 0.37506 & 0 & & & \\ -0.37506 & 0.927 & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hline & & & 0.927 & 0.37506 & 0 \\ & 0 & & -0.37506 & 0.927 & 0 \\ & & & 0 & 0 & 1 \end{array}\right]$$

$$K_{Gbc} = T^T K'_{bc} T$$

$$K_{gbc} = 200 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 0 & 4 & \end{array} \\ \begin{bmatrix} 0.6452 & 0.2591 & -7.0313 & -0.6452 & -0.2591 & -7.0313 \\ 0.2591 & 0.1095 & 17.381 & -0.2591 & -0.1095 & 17.381 \\ -7.0313 & 17.381 & 100{,}000 & 7.0313 & -17.381 & 50{,}000 \\ -0.6452 & -0.2591 & 7.0313 & 0.06452 & 0.2591 & 7.0313 \\ -0.2591 & -0.1095 & -17.381 & 0.2591 & 0.1095 & -17.381 \\ -7.0313 & 17.381 & 50{,}000 & 7.0313 & -17.381 & 100{,}000 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 0 \\ 4 \\ \\ \end{array}$$

$$DA = \begin{Bmatrix} 1 \\ 2 \\ 3 \\ 0 \\ 4 \\ 0 \end{Bmatrix}$$

Symmetric Portion

$$K_{Struc} = 200 \begin{bmatrix} 0.0048 + 0.6452 & 0 + 0.2591 & 12 - 7.0313 & -0.2591 \\ 0 & 0.8 + 0.1095 & 0 + 17.381 & -0.1095 \\ & 0 & 40{,}000 + 100{,}000 & -17.381 \\ & \text{Sym} & & 0.1095 \end{bmatrix}$$

$$200 \begin{bmatrix} 0.65 & 0.2591 & 4.969 & -0.2591 \\ & 0.9095 & 17.381 & -0.1095 \\ & & 140000 & -17.381 \\ & \text{sym} & & 0.1095 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \theta_3 \\ \Delta_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -5 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \theta_3 \\ \Delta_4 \end{Bmatrix} = \begin{Bmatrix} -2.208 \\ -0.0313 \\ -0.611 \times 10^{-3} \text{ rad} \\ -5.583 \end{Bmatrix}$$

Symmetric Part

Matrix Analysis – Stiffness Method

Non Symmetric Part

Stiffness assembly yields the following stiffness matrix and Equilibrium equations

$$200\begin{bmatrix} 0.65 & 0.2591 & 4.969 & -0.6452 & -7.0313 \\ & 0.9095 & 17.381 & -0.2591 & 17.381 \\ & & 140{,}000 & 7.0313 & 50{,}000 \\ & & & 0.6452 & 7.0313 \\ & & & & 100{,}000 \end{bmatrix} \begin{Bmatrix}\Delta_1\\ \Delta_2\\ \theta_3\\ \Delta_4\\ \theta_5\end{Bmatrix} = \begin{Bmatrix}5\\0\\0\\0\\0\end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix}\Delta_1\\ \Delta_2\\ \theta_3\\ \Delta_4\\ \theta_5\end{Bmatrix} = \begin{Bmatrix}7.085\\ 0.0093\\ -0.704\times10^{-3}\\ 7.093\\ 0.368\times10^{-3}\end{Bmatrix} \begin{matrix}\text{mm}\\ \text{mm}\\ \text{rad}\\ \text{mm}\\ \text{rad}\end{matrix}$$

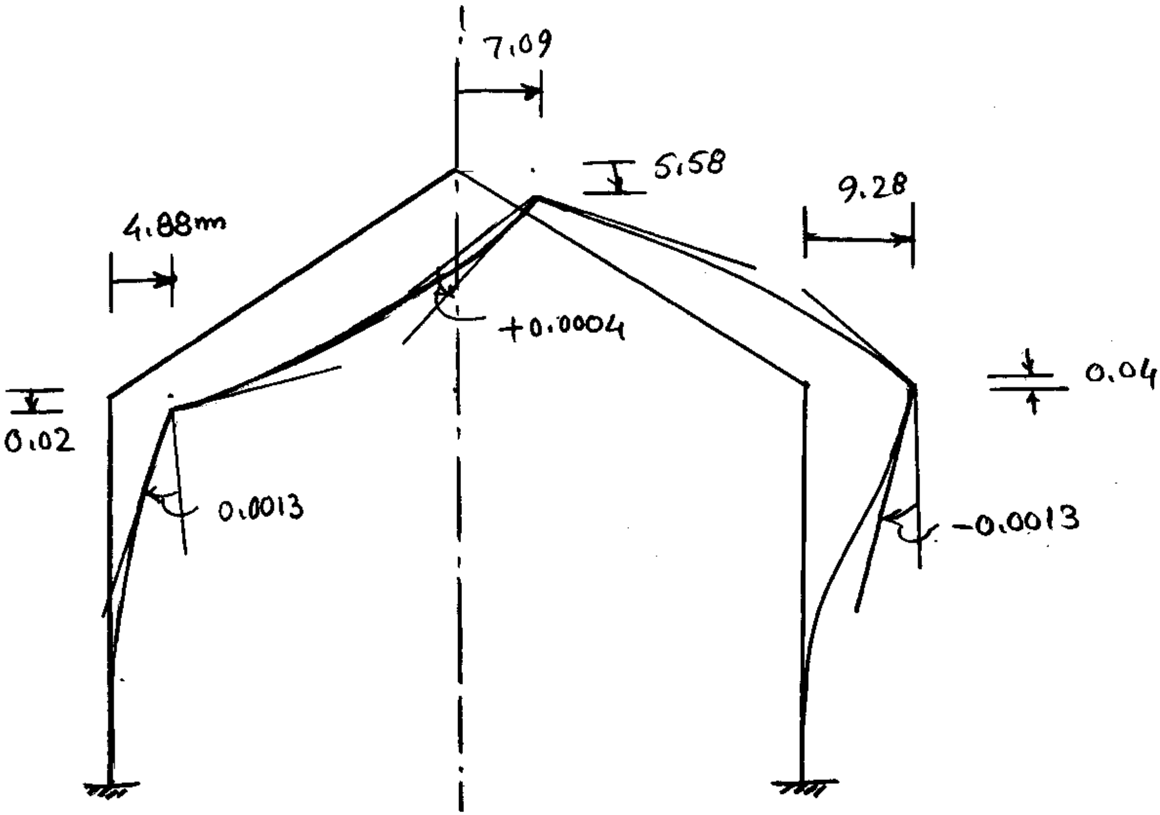
Antisymmetric Part

$$\begin{Bmatrix}\Delta_1\\ \Delta_2\\ \theta_3\\ \Delta_4\\ \Delta_5\\ \theta_6\end{Bmatrix} = \underbrace{\begin{Bmatrix}-2.208\\ -0.0313\\ -0.611\times10^{-3}\\ 0\\ -5.583\\ 0\end{Bmatrix}}_{\text{Symmetric}} + \underbrace{\begin{Bmatrix}7.085\\ 0.0093\\ -0.704\times10^{-3}\\ 7.093\\ 0.368\times10^{-3}\end{Bmatrix}}_{\text{Antisymmetric}}$$

$$\begin{Bmatrix}\Delta_1\\ \Delta_2\\ \theta_3\\ \Delta_4\\ \theta_6\end{Bmatrix} = \begin{Bmatrix}4.88\\ -0.02\\ -1.351\times10^{-3}\\ 7.093\\ 0.368\times10^{-3}\end{Bmatrix} \begin{matrix}\text{mm}\\ \text{mm}\\ \text{rad}\\ \text{mm}\\ \text{rad}\end{matrix}$$

Solution

Matrix Analysis – Stiffness Method



Structure Deflected Shape