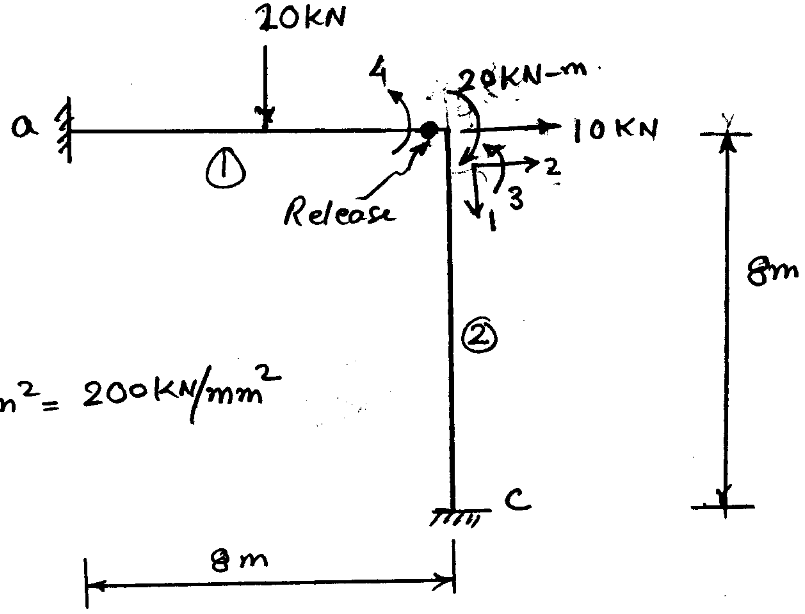
Matrix Analysis – Member Releases

Example Problem

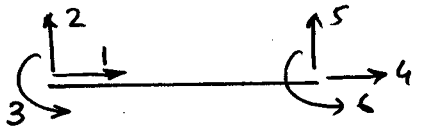
Analyze the Frame shown below in which member ab has a moment release at end b



$A = 6000\,mm^2$

$I = 200 \times 10^6\,mm^4$

$E = 200{,}000\,N/mm^2 = 200\,KN/mm^2$



$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ F_4 \\ F_5 \\ M_6 \end{Bmatrix} = E \begin{bmatrix} A/L & 0 & 0 & -\frac{A}{L} & 0 & 0 \\ 0 & \frac{12I}{L^3} & \frac{6I}{L^2} & 0 & -\frac{12I}{L^3} & \frac{6I}{L^2} \\ 0 & \frac{6I}{L^2} & \frac{4I}{L} & 0 & -\frac{6I}{L^2} & \frac{2I}{L} \\ -\frac{A}{L} & 0 & 0 & \frac{A}{L} & 0 & 0 \\ 0 & -\frac{12}{L^3} & -\frac{6I}{L^2} & 0 & \frac{12I}{L^3} & -\frac{6I}{L^2} \\ 0 & \frac{6}{L^2} & \frac{2I}{L} & 0 & -\frac{6I}{L^2} & \frac{4I}{L} \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \theta_3 \\ \Delta_4 \\ \Delta_5 \\ \theta_6 \end{Bmatrix}$$

Example Problem

Member ①

$$\frac{A}{L} = \frac{6000}{8000} = 0.75$$

$$\frac{12I}{L^3} = \frac{12\times 200\times 10^6}{(8000)^3} = 4.6875\times 10^{-3}\ mm$$
$$= 0.00469\ mm.$$

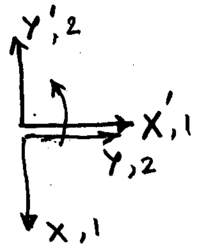
$$\frac{6I}{L^2} = \quad = 18.75$$

$$\frac{4I}{L} = \quad = 100,000$$

$$\frac{2I}{L} = \quad = 50,0000$$

K

$$K_1 = 200 \left[\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ \hline & & & 0.75 & 0 & 0 \\ & & & 0 & 0.00469 & -18.75 \\ & & & 0 & -18.75 & 100,000 \end{array}\right]$$



$$K_G = T^T K_{①} T$$

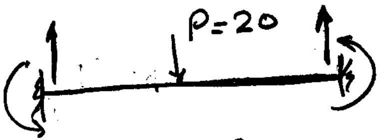
$$[\gamma] = \begin{array}{c|ccc} & x & Y & z \\ \hline x' & 0 & 1 & 0 \\ y' & -1 & 0 & 0 \\ z' & 0 & 0 & 1 \end{array}$$

$$KG_{①} = 200 \left[\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ \hline & & & 0 & -1 & 0 \\ & & & 1 & 0 & 0 \\ & & & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ \hline & & & 0.75 & 0 & 0 \\ & & & 0 & .00469 & -18.75 \\ & & & 0 & -18.75 & 100,000 \end{array}\right]$$

$$\times \left[\begin{array}{ccc|ccc} & & & & & \\ \hline & & & 0 & 1 & 0 \\ & & & -1 & 0 & 0 \\ & & & 0 & 0 & 1 \end{array}\right]$$

Example Problem

$$KG_{①} = 200 \left[\begin{array}{c|ccc|c} & 1 & 2 & 4 & \\ \hline \quad & & & & \\ \hline & 0.00469 & 0 & 18.75 & 2 \\ & 0 & 0.75 & 0 & 1 \\ & 18.75 & 0 & 100{,}000 & 4 \end{array}\right] \qquad DA_{①} = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 1 \\ 2 \\ 4 \end{array}\right\}$$



Fr

$$FEM = \left\{\begin{array}{c} 0 \\ P/2 \\ \frac{PL}{8} \\ \hline 0 \\ P/2 \\ -\frac{PL}{8} \end{array}\right\} = \left\{\begin{array}{c} 0 \\ 10 \\ 20 \\ \hline 0 \\ 10 \\ -20 \end{array}\right\} \begin{array}{l} \\ \\ \\ \\ KN \\ KN\text{-}m \end{array} = \left\{\begin{array}{c} 0 \\ 10 \\ 20{,}000 \\ \hline 0 \\ 10 \\ -20{,}000 \end{array}\right\} \begin{array}{l} \\ \\ KN\text{-}mm \\ \\ KN \\ KN\text{-}mm \end{array}$$

$$FEQ = -FEM = \left\{\begin{array}{c} 0 \\ -10 \\ -20{,}000 \\ \hline 0 \\ -10 \\ -20{,}000 \end{array}\right\}$$

$$F_{EQ}\ Global = [T]^T \{F_{EQ}\}$$

$$= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{\begin{array}{c} 0 \\ -10 \\ 20{,}000 \end{array}\right\} = \left\{\begin{array}{c} 10 \\ 0 \\ 20{,}000 \end{array}\right\}$$

$$\Rightarrow F_{EQ\,Global\ ①} = \left\{\begin{array}{c} -- \\ -- \\ --- \\ \hline 10 \\ 0 \\ 20{,}000 \end{array}\right\} \qquad DA_1 = \left\{\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 1 \\ 2 \\ 4 \end{array}\right\}$$

Example Problem

Member 2

$$K_{②\,local} = K_{Global} = 200\left[\begin{array}{ccc|c} 0.75 & 0 & 0 & \\ 0 & 0.00469 & 18.75 & \\ 0 & 18.75 & 100{,}000 & \\ \hdashline & & & \\ & & & \\ & & & \end{array}\right] \qquad DA_{②} = \left\{\begin{array}{c} 1 \\ 2 \\ 3 \\ \hdashline 0 \\ 0 \\ 0 \end{array}\right\}$$

Structural Stiffness Matrix

$$K_G = 200\left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 0.00469 + 0.75 & 0 & 0 & 18.75 \\ 0 & 0.75 + 0.00469 & 0 \; 18.75 & 100 \\ 0 & 0 + 18.75 & 0 \; 100{,}000 & 0 \; 100{,}000 \\ 18.75 & 100 & 0 & \end{array}\right]$$

$$K_G = 200\left[\begin{array}{cccc} 0.75469 & 0 & 0 & 18.75 \\ 0 & 0.75469 & 18.75 & 0 \\ 0 & 18.75 & 100{,}000 & 0 \\ 18.75 & 0 & 0 & 100{,}000 \end{array}\right]$$

$$F_{EQ\,Global} = \left\{\begin{array}{c} 10 + \\ 0 + 10 \\ 0 - 20{,}000 \\ 20000 \end{array}\right\} = \left\{\begin{array}{c} 10 \\ 10 \\ -20{,}000 \\ 20{,}000 \end{array}\right\} \begin{array}{l} KN \\ KN. \\ KN\text{-}m \\ KN\text{-}mm \end{array}$$

Example Problem

$$[KG]\ \{\Delta\} = \{P_{EQ}\}$$

$$200\left[\begin{array}{ccc|c} 0.75469 & 0 & 0 & 18.75 \\ 0 & 0.75469 & 18.75 & 0 \\ 0 & 18.75 & 100{,}000 & 0 \\ \hline 18.75 & 0 & 0 & 100{,}000 \end{array}\right] \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} 10 \\ 10 \\ -20{,}000 \\ 20{,}000 \end{Bmatrix}$$

Direct Soln gives

$$\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \frac{1}{200} \begin{Bmatrix} 8.3203 \\ 18.30467 \\ -0.203432 \\ 0.1984 \end{Bmatrix} = \begin{Bmatrix} 0.416 \\ 0.0915 \\ -0.001017 \\ 0.00099 \end{Bmatrix} \begin{matrix} \text{mm} \\ \text{mm} \\ \text{rad} \\ \text{rad.} \end{matrix}$$

Alternately Condense out θ_4 From Global Equations

$$\left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \left\{\begin{array}{c} \Delta_r \\ \hline \Delta_c \end{array}\right\} = \left\{\begin{array}{c} P_r \\ \hline P_c \end{array}\right\}$$

$$K_{21}\,\Delta_r + K_{22}\,\Delta_c = P_c$$

$$\Rightarrow \quad \Delta_c = K_{22}^{-1}\left(P_c - K_{21}\,\Delta_r\right)$$

$$K_{11}\,\Delta_r + K_{12}\,\Delta_c = P_r$$

$$K_{11}\,\Delta_r + K_{12}\left(K_{22}^{-1}\left(P_c - K_{21}\,\Delta_r\right)\right) = P_r$$

$$\boxed{\left[K_{11} - K_{12}K_{22}^{-1}K_{21}\right]\Delta_r = P_r - K_{12}\,K_{22}^{-1}\,P_c}$$

Condensed Equations

Example Problem

$$K_{12}\,K_{22}^{-1} = 200 \begin{Bmatrix} 18.75 \\ 0 \\ 0 \end{Bmatrix} \times \left[\frac{1}{200 \times 100{,}000}\right]$$

$$= \frac{1}{100{,}000} \begin{Bmatrix} 18.75 \\ 0 \\ 0 \end{Bmatrix}$$

$$K_{12}\,K_{22}^{-1}\,K_{21} = \frac{200}{100{,}000} \begin{bmatrix} 18.75 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 18.75 & 0 & 0 \end{bmatrix}$$

$$= \frac{200}{100{,}000} \begin{bmatrix} 851.563 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\tilde{K}] = K_{11} - K_{12}\,K_{22}^{-1}\,K_{21}$$

$$[\tilde{K}] = 200 \begin{bmatrix} 0.75117 & 0 & 0 \\ 0 & 0.75469 & 18.75 \\ 0 & 18.75 & 100{,}000 \end{bmatrix}$$

$$[\tilde{P}] = P_{r} - K_{12}\,K_{22}^{-1}\,P_{c}$$

$$= \begin{Bmatrix} 10 \\ 10 \\ -20{,}000 \end{Bmatrix} - \frac{1}{100{,}000} \begin{Bmatrix} 18.75 \\ 0 \\ 0 \end{Bmatrix} \left[20{,}000\right] = \begin{Bmatrix} 6.25 \\ 10 \\ -20{,}000 \end{Bmatrix}$$

Example Problem

$$[\tilde{K}]\{\Delta_r\} = \{\tilde{P}_r\}$$

$$\Rightarrow \{\Delta_r\} = [\tilde{K}]^{-1}\{P_r\}$$

$$\Rightarrow \{\Delta_r\} = \begin{Bmatrix}\Delta_1\\ \Delta_2\\ \theta_3\end{Bmatrix} = \frac{1}{200}\begin{Bmatrix}8.3204\\ 18.304\\ -0.2034\end{Bmatrix} = \begin{Bmatrix}0.0416\\ 0.0915\\ -0.001017\end{Bmatrix}$$

Recovering the Condensed DOF

$$\Delta_c = \theta_4 = k_{22}^{-1}\left(P_c - K_{21}\Delta_r\right)$$

$$= \frac{1}{200 \times 100{,}000}\left[[20{,}000] - \frac{1200}{200}\begin{bmatrix}18.75 & 0 & 0\end{bmatrix}\begin{Bmatrix}8.3204\\ 18.304\\ -0.2034\end{Bmatrix}\right]$$

$$\Delta_c = \theta_4 = \frac{1}{200 \times 100{,}000}\left[19843.99\right] = \frac{1}{200} \times 0.1984 =$$

$$= 0.00099 \text{ rad.}$$

Hence the Complete Displacement Vector is again

$$\boxed{\begin{Bmatrix}\Delta_1\\ \Delta_2\\ \theta_3\\ \theta_4\end{Bmatrix} = \frac{1}{200}\begin{Bmatrix}8.303\\ 18.30467\\ -0.203432\\ 0.1984\end{Bmatrix} = \begin{Bmatrix}0.416\\ 0.0915\\ -0.001017\\ 0.00099\end{Bmatrix}\begin{matrix}\text{mm}\\ \text{mm}\\ \text{rad}\\ \text{rad.}\end{matrix}}$$

Example Problem

As a Second Alternate we can condense out θ_4 at Element Level when processing Element ①

Recall that For Element ① we have the Equilibrium Equations in Global Coordinates as:

$$200 \left[\begin{array}{cc|c} 0.00469 & 0 & 18.75 \\ 0 & 0.75 & 0 \\ \hline 18.75 & 0 & 100,000 \end{array}\right] \left\{\begin{array}{c} \Delta_1 \\ \Delta_2 \\ \hline \theta_4 \end{array}\right\} = \left\{\begin{array}{c} 10 \\ 0 \\ \hline 20,000 \end{array}\right\}$$

$$\underset{①}{KG} \qquad\qquad \underset{①}{FEQ\,G}$$

Condense out θ_4

$$K_{12} K_{22}^{-1} = 200 \begin{bmatrix} 18.75 \\ 0 \end{bmatrix} \left[\frac{1}{209 \times 100,000} \right] = 200 \begin{bmatrix} 9.375 \times 10^{-7} \\ 0 \end{bmatrix}$$

$$K_{12} K_{22}^{-1} K_{21} = 200 \begin{bmatrix} 9.375 \times 10^{-7} \\ 0 \end{bmatrix} \left[18.75 \;|\; 0 \right]$$

$$K_{12} K_{22}^{-1} K_{21} = 200 \begin{bmatrix} 1.7578 \times 10^{-5} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\tilde{K}G_{①} = K_{11} - K_{12} K_{22}^{-1} K_{21}$$

$$\tilde{K}G_{①} = 200 \begin{bmatrix} 0.00469 & 0 \\ 0 & 0.75 \end{bmatrix} - 200 \begin{bmatrix} 1.7578 \times 10^{-5} & 0 \\ 0 & 0 \end{bmatrix}$$

$$\tilde{K}G_{①} = 200 \begin{bmatrix} 0.004672 & 0 \\ 0 & 0.75 \end{bmatrix} \qquad \tilde{D}A_{①} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

Example Problem

$$\tilde{P}_{G①} = P_r - K_{12} K_{22}^{-1} P_c$$

$$= \begin{Bmatrix} 10 \\ 0 \end{Bmatrix} - 200 \begin{bmatrix} 9.375\times10^{-7} \\ 0 \end{bmatrix} \begin{bmatrix} 20,000 \end{bmatrix}$$

$$\tilde{P}_{G①} = \begin{Bmatrix} 6.25 \\ 0 \end{Bmatrix} \qquad \tilde{DA}_{①} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

Assembled Structure Stiffness Matrix

$$\tilde{K}_G = 200 \left[\begin{array}{ccc} 1 & 2 & 3 \\ 0.004672 + 0.75 & 0 & 0 \\ 0 & 0.75 + 0.00469 & 0 + 18.75 \\ 0 & 0 + 18.75 & 0 + 100,000 \end{array}\right]$$

$$\tilde{K}_G = 200 \begin{bmatrix} 0.754672 & 0 & 0 \\ 0 & 0.754672 & 18.75 \\ 0 & 18.75 & 100,000 \end{bmatrix}$$

$$\tilde{P}_{EQ} = \begin{Bmatrix} 6.25 + 0 \\ 0 + 10 \\ 0 - 20,000 \end{Bmatrix} = \begin{Bmatrix} 6.25 \\ 10 \\ -20,000 \end{Bmatrix}$$

$$\tilde{K}_G \tilde{\Delta}_r = \tilde{P}_{EQ}$$

$$\Rightarrow \tilde{\Delta}_r = \tilde{K}_G^{-1} P_{EQ}$$

$$\Rightarrow \tilde{\Delta}_r = \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \theta_3 \end{Bmatrix} = \frac{1}{200} \begin{Bmatrix} 8.2817 \\ 18.305 \\ -0.2034 \end{Bmatrix} = \begin{Bmatrix} 0.416 \\ 0.0915 \\ -0.001017 \end{Bmatrix} \begin{matrix} \text{mm} \\ \text{mm} \\ \text{rad} \end{matrix}$$

Same as before

Example Problem

Hence, we have demonstrated that we can provide a force release in the following number of manners:

- Introduce an extra Degree of freedom and solving directly
- Assemble Global Structure Equilibrium Equations and Condense out the Degree of Freedom corresponding to the Released Force
- Condense out the Degree of Freedom corresponding to Released Force at Element Formulation Level and then proceed with Stiffness Assembly

Determination of Member Forces

$$\{F\} = [K]\{\Delta\} + FEM.$$

@ Fixed End Member ①

$$= \frac{200}{200}\begin{bmatrix} -0.75 & 0 & 0 \\ 0 & -0.00469 & 18.75 \\ 0 & -18.75 & 50,000 \end{bmatrix}\begin{Bmatrix} -8.303 \\ 18.30467 \\ 0.1984 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 10 \\ 20,000 \end{Bmatrix}$$

$$= \begin{Bmatrix} +6.227 \\ +3.634 \\ +9576.8 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 10 \\ 20,000 \end{Bmatrix} = \begin{Bmatrix} 6.227 \\ 13,634 \\ 29576.8 \end{Bmatrix}\begin{matrix} KN \\ KN \\ KN\text{-}mm \end{matrix} = \begin{Bmatrix} 6.23 \\ 13.63 \\ 29.58 \end{Bmatrix}\begin{matrix} KN \\ KN. \\ KN\text{-}m \end{matrix}$$

@ Hinged End Member ①

$$= \frac{200}{200}\begin{bmatrix} 0.75 & 0 & 0 \\ 0 & 0.00469 & -18.75 \\ 0 & -18.75 & 100,000 \end{bmatrix}\begin{Bmatrix} -8.303 \\ 18.30467 \\ 0.1984 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 10 \\ -20,000 \end{Bmatrix}$$

$$= \begin{Bmatrix} -6.227 \\ -3.634 \\ +19496.8 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 10 \\ -20,000 \end{Bmatrix} = \begin{Bmatrix} -6.227 \\ +6.366 \\ -503 \end{Bmatrix}\begin{matrix} KN \\ KN \\ KN\text{-}mm \end{matrix} = \begin{Bmatrix} -6.227 \\ +6.366 \\ -0.5 \end{Bmatrix}\begin{matrix} KN \\ KN \\ KN\text{-}m \end{matrix}$$

Example Problem

Forces in Member ②

Top End

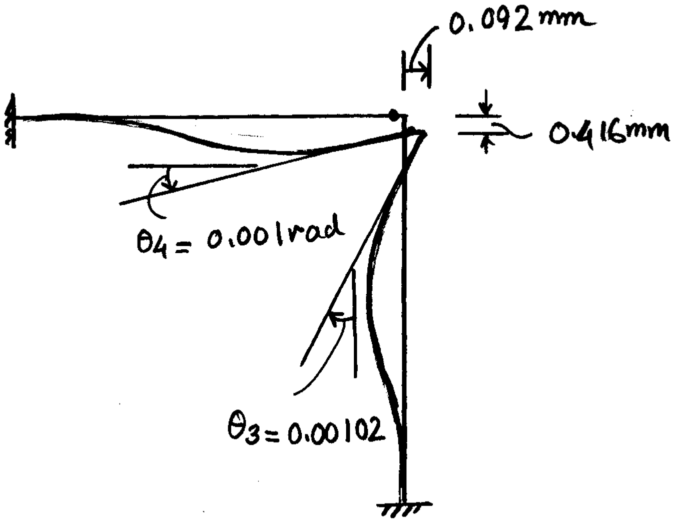
$$\frac{200}{200}\begin{bmatrix} 0.75 & 0 & 0 \\ 0 & 0.00469 & 18.75 \\ 0 & 18.75 & 100{,}000 \end{bmatrix}\begin{Bmatrix} 8.3203 \\ 18.30467 \\ -0.1984 \end{Bmatrix} = \begin{Bmatrix} 6.24 \\ -3.63 \\ -19496 \end{Bmatrix}\begin{matrix} \text{KN} \\ \text{KN} \\ \text{KN-mm} \end{matrix}$$

$$= \begin{Bmatrix} 6.24 \\ -3.63 \\ -19.5 \end{Bmatrix}\begin{matrix} \text{KN} \\ \text{KN.} \\ \text{KN-m} \end{matrix}$$

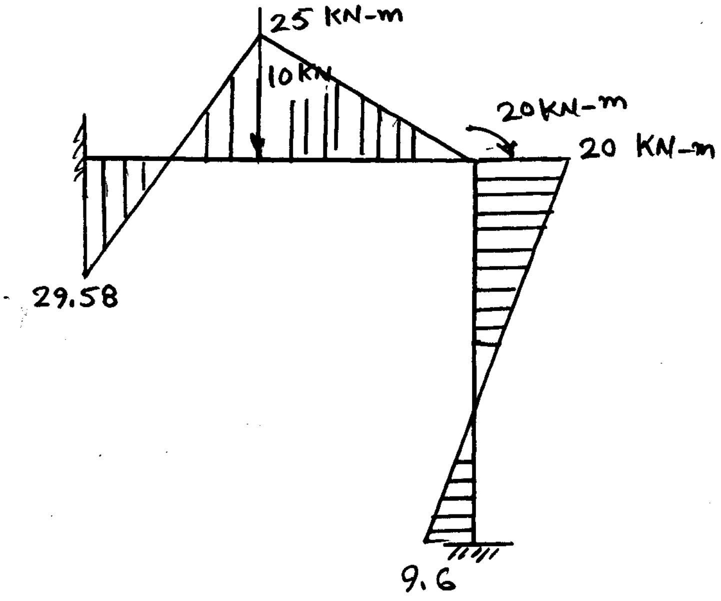
Bottom End

$$\frac{200}{200}\begin{bmatrix} -0.75 & 0 & 0 \\ 0 & -0.00469 & -18.75 \\ 0 & 18.75 & 50{,}000 \end{bmatrix}\begin{Bmatrix} 8.3203 \\ 18.30467 \\ -0.1984 \end{Bmatrix} = \begin{Bmatrix} -6.24 \\ +3.63 \\ -9.567 \end{Bmatrix}\begin{matrix} \text{KN} \\ \text{KN} \\ \text{KN-mm} \end{matrix}$$

$$= \begin{Bmatrix} -6.24 \\ +3.63 \\ -9.576 \end{Bmatrix}$$



DFFLECTED SHAPE



BENDING MOMENT