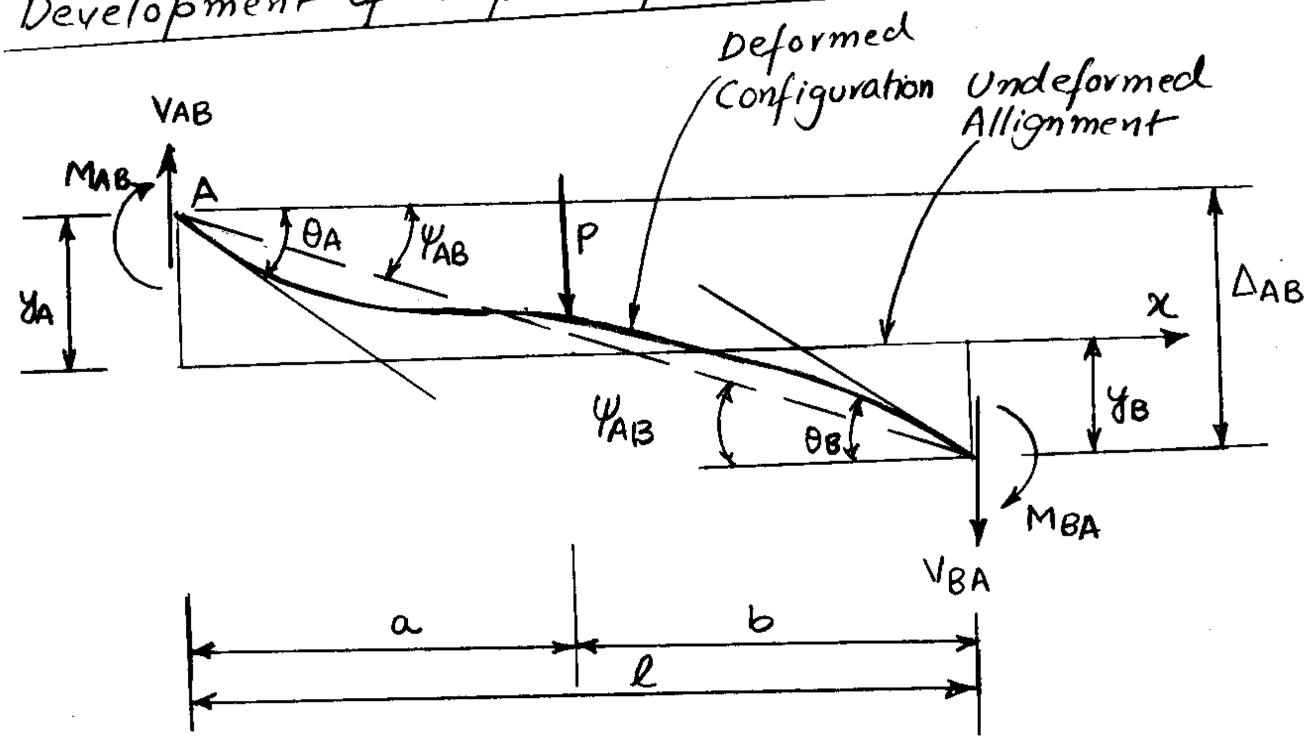
SLOPE DEFLECTION METHOD

- Slope Deflection Method was presented by G. A. Maney in 1915 as a method of analysis for rigid-jointed beam and frame structures.
- It is an Equilibrium based method. Equilibrium methods are based on solution of equilibrium equations for the entire structural system. in which one equilibrium equation is written for each kinematic degree of freedom while maintaining conditions of compatibility and the boundary conditions.

Development of Slope-Deflection Equations



Boundary Conditions

$$@\ Y = 0 \qquad Y\Big|_{x=0} = Y_A \ , \quad \frac{dy}{dx}\Big|_{x=0} = -\theta_A \qquad \text{— (1)}$$

$$@\ Y = l \qquad Y\Big|_{x=l} = -Y_B \ , \quad \frac{dy}{dx}\Big|_{x=l} = -\theta_B$$

SLOPE-DEFLECTION METHOD

$$M = EI \frac{d^2y}{dx^2} \qquad \text{—— (2)}$$

$$M = M_{AB} + V_{AB}\, x - P\{x - a\} \qquad \text{—— (3)}$$

Where,

$\{x - a\} = 0$ if $x - a$ is -ive

Substituting eq (3) in eq (2) we get

$$\frac{d^2y}{dx^2} = \frac{M_{AB}}{EI} + \frac{V_{AB}}{EI}\, x - \frac{P}{EI}\{x - a\} \qquad \text{—— (4)}$$

Integrating eqn (4) and using Boundary conditions eqns (1) we get

$$\frac{dy}{dx} = \frac{M_{AB}\, x}{EI} + \frac{V_{AB}}{2EI}\, x^2 - \frac{P}{2EI}\{x - a\}^2 - \theta_A$$

$$Y = \frac{M_{AB}\, x^2}{2EI} + \frac{V_{AB}}{6EI}\, x^3 - \frac{P}{6EI}\{x - a\}^3 - \theta_A \cdot x + Y_A \qquad \text{—— (5)}$$

Applying Boundary Conditions at $x = l$ we have

$$-\theta_B = \frac{M_{AB}\, l}{2EI} + \frac{V_{AB}}{2EI}\, l^2 - \frac{P}{2EI}\{l - a\}^2 - \theta_A$$

$$-Y_B = \frac{M_{AB}\, l^2}{2EI} + \frac{V_{AB}\, l^3}{6EI} - \frac{P}{2EI}\{l - a\}^3 - \theta_A\, l + Y_A \qquad \text{—— (6)}$$

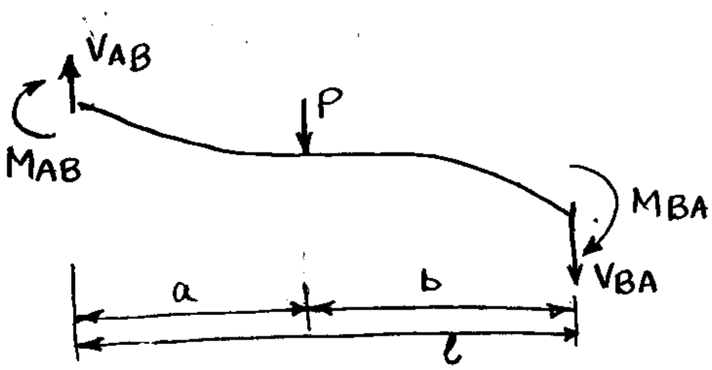
SLOPE-DEFLECTION METHOD

Simultaneously solving eqns (6) for M_{AB} and V_{AB} we have

$$M_{AB} = \frac{2EI}{\ell}\left(2\theta_A + \theta_B - \frac{3y_A}{\ell} - \frac{3y_B}{\ell}\right) - \frac{Pab^2}{\ell^2} \quad \text{— (7)}$$

$$V_{AB} = \frac{6EI}{\ell^2}\left(-\theta_A - \theta_B + \frac{2y_A}{\ell} + \frac{2y_B}{\ell}\right) + \frac{Pb^2(\ell+2a)}{\ell^3}$$

By application of Statics we have



$$V_{AB} - P - V_{BA} = 0 \longrightarrow$$

Also

$$\boxed{\begin{aligned} V_{BA} &= V_{AB} - P \\ M_{BA} &= M_{AB} + V_{AB}\ell - Pb \end{aligned}} \quad \text{— (8)}$$

Substituting M_{AB} and V_{AB} from eq (7) into eqs (8) and simplifying results yields the following expressions

$$\boxed{\begin{aligned} M_{BA} &= \frac{2EI}{\ell}\left(2\theta_B + \theta_A - \frac{3y_A}{\ell} - \frac{3y_B}{\ell}\right) + \frac{Pa^2b}{\ell^2} \\ V_{BA} &= -\frac{6EI}{\ell^2}\left(\theta_A + \theta_B - \frac{2y_A}{\ell} - \frac{2y_B}{\ell}\right) - \frac{Pa^2(\ell+2b)}{\ell^3} \end{aligned}} \quad \text{— (9)}$$

SLOPE-DEFLECTION METHOD

Arranging eqns 7 and 9 in matrix form we have

$$\boxed{\begin{Bmatrix} M_{AB} \\ V_{AB} \\ M_{BA} \\ V_{BA} \end{Bmatrix} = \frac{2EI}{l}\begin{bmatrix} 2 & -3/l & 1 & -3/l \\ -3/l & 6/l^2 & -3/l & 6/l^2 \\ 1 & -3/l & 2 & -3/l \\ -3/l & 6/l^2 & -3/l & 6/l^2 \end{bmatrix}\begin{Bmatrix} \theta_A \\ y_A \\ \theta_B \\ y_B \end{Bmatrix} + \begin{Bmatrix} -Pab^2/l \\ Pb^2(l+2a)/l^3 \\ Pa^2b/l \\ -Pa^2(l+2b)/l^3 \end{Bmatrix}} \quad \text{—— (10)}$$

Writing above eqn in further short form we have:

$$\boxed{\{F\} = [K]\{\delta\} + \{F\}^f} \quad \text{—— (11)}$$

$\{F\}$ = Member End Forces Vector

$[K]$ = Member Stiffness Matrix

$\{\delta\}$ = Member End Displacement Vector

$\{F\}^f$ = Member Fixed End Forces

Note if Member End Displacement Vector $\{\delta\} = \{0\}$

Then $\{F\} = \{F\}^f$

The expressions for end moments of a beam under general loading may be written as follows:

$$\boxed{\begin{aligned} M_{AB} &= \frac{2EI}{l}\left(2\theta_A + \theta_B - 3\psi_{AB}\right) + FEM_{AB} \\ M_{BA} &= \frac{2EI}{l}\left(2\theta_B + \theta_A - 3\psi_{AB}\right) + FEM_{BA} \end{aligned}} \quad \text{—— (12)}$$

FEM_{AB}, FEM_{BA} = Fixed End Moments

ψ_{AB} = Rotation of Chord AB

$$\boxed{\psi_{AB} = \frac{\Delta_{AB}}{l} = \frac{y_B + y_A}{l}} \quad \text{—— (13)}$$

Eqs (12) are referred to as the Slope Deflection Equations.

These can be further generalized as a single equation

$$\boxed{M_{nf} = 2E\,K_{nf}\left(2\theta_n + \theta_f - 3\psi_{nf}\right) + FEM_{nf}} \quad \text{—— (14)}$$

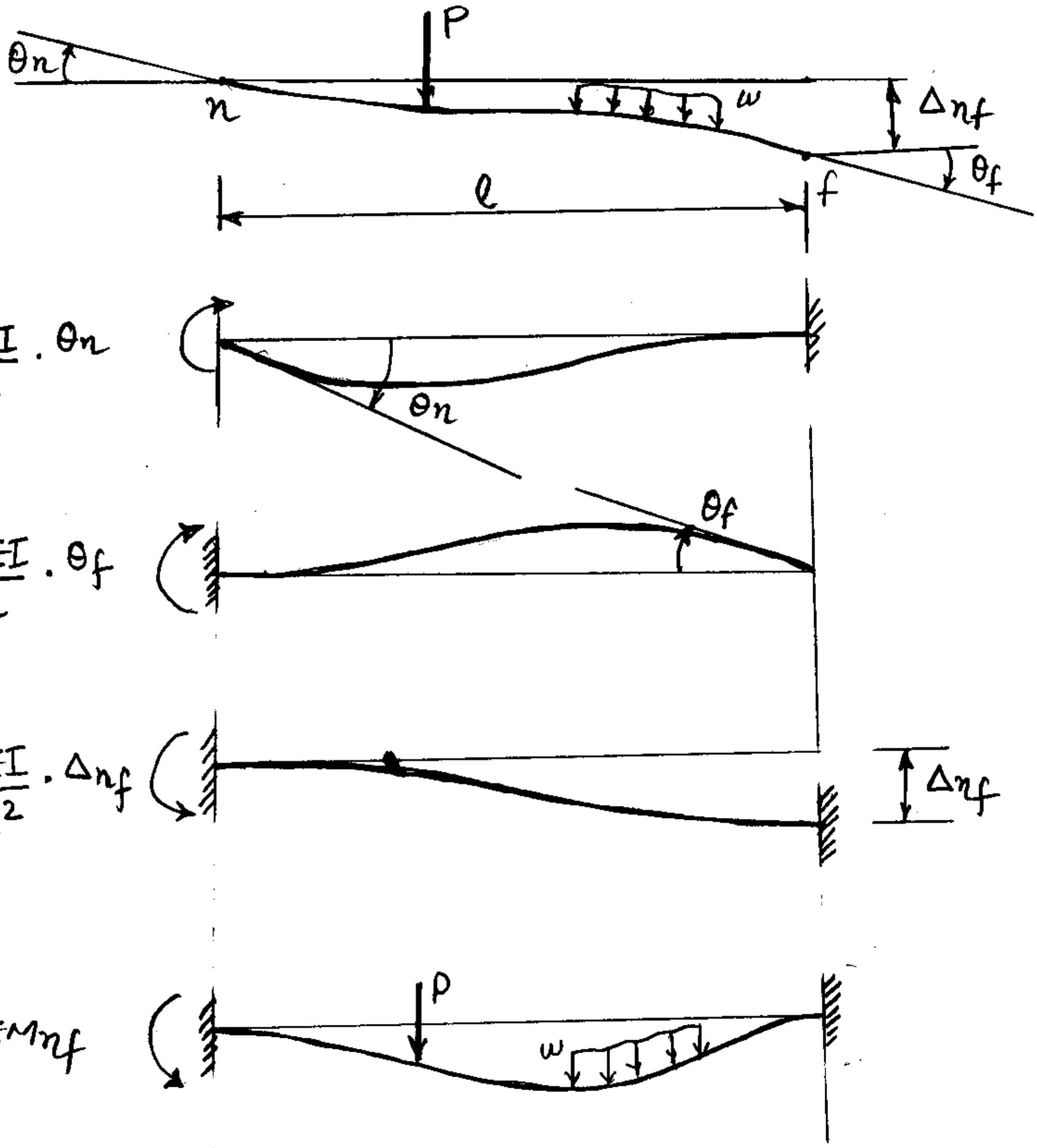
Subscripts n and f refer to the near and far ends of the member.

$$\boxed{K_{nf} = \text{Member Stiffnes Factor} = \frac{I}{l}} \quad \text{—— (15)}$$

SLOPE DEFLECTION METHOD

$$M_{nf} = \frac{4EI}{l}\theta_n + \frac{2EI}{l}\theta_f - \frac{6EI}{l^2}\Delta_{nf} + FEM_{nf} \quad \text{--- 15}$$

The Figure below illustrates the physical meaning of each of the four terms in the above expanded Slope-Deflection Equation.



$M = \frac{4EI}{l} . \theta_n$

+

$M = \frac{2EI}{l} . \theta_f$

+

$M = \frac{6EI}{l^2} . \Delta_{nf}$

+

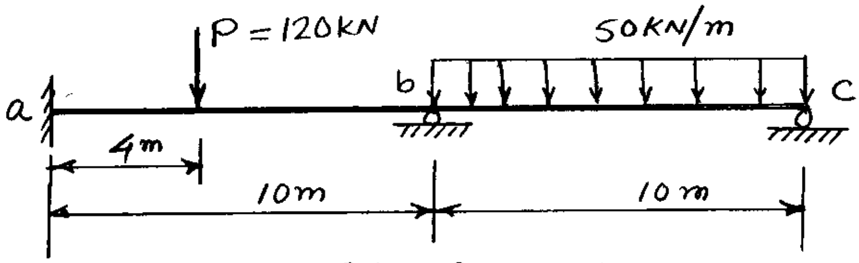
$M = FEM_{nf}$

$$\longrightarrow M_{nf} = \frac{4EI}{l}\theta_n + \frac{2EI}{l} . \theta_f - \frac{6EI}{l^2}\Delta_{n} +$$

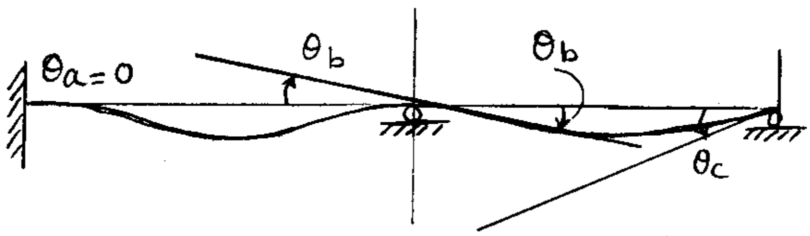
SLOPE DEFLECTION METHOD

Example

Determine the end moments and shearforce and bending moment diagrams for the beam shown below.



EI = Constant



Compatibility & Boundary Conditions

$$M_{nf} = 2EK_{nf}(2\theta_n + \theta_f - 3\psi_{nf}) + FEM_{nf}$$

$$K_{ab} = K_{bc} = I/L = I/10 = K$$

$$FEM_{ab} = -\frac{Pab^2}{l^2} = -\frac{120 \times 4 \times 6^2}{10^2} = -172.8 \text{ KN-m}$$

$$FEM_{ba} = \frac{Pba^2}{l^2} = \frac{120 \times 6 \times 4^2}{10^2} = +115.2 \text{ KN-m}$$

$$FEM_{bc} = -\frac{wl^2}{12} = -\frac{50 \times 10^2}{12} = -416.7 \text{ KN-m}$$

$$FEM_{cb} = \frac{wl^2}{12} = \frac{50 \times 10^2}{12} = +416.7 \text{ KN-m}$$

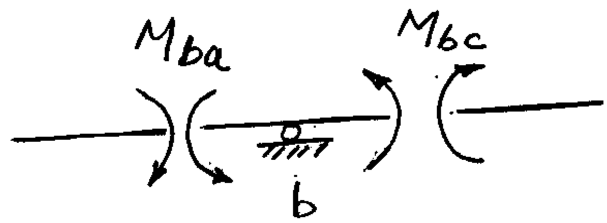
$$\left.\begin{aligned}
M_{ab} &= 2EK\,\theta_b - 172.8 \\
M_{ba} &= 2EK(2\theta_b) + 115.2 \\
M_{bc} &= 2EK(2\theta_b + \theta_c) - 416.7 \\
M_{cb} &= 2EK(2\theta_c + \theta_b) + 416.7
\end{aligned}\right\} \quad \text{—(1)}$$

Slope Deflection Equations.

Example – Slope Deflection Method

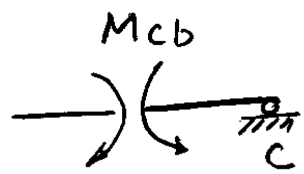
Equilibrium Conditions

At Joint b



$$M_{ba} + M_{bc} = 0$$

At Joint c



$$M_{cb} = 0$$

$$\boxed{\begin{aligned} M_{ba} + M_{bc} &= 0 \\ M_{cb} &= 0 \end{aligned}} \quad \text{—— (2)}$$

Substituting the Slope Deflection Equations (1) into the compatibility equations (2) we have

$$4EK\theta_b + 115.2 + 4EK\theta_b + 2EK\theta_c - 416.7 = 0$$

$$2EK\theta_b + 4EK\theta_c + 416.7 = 0$$

$$8EK\theta_b + 2EK\theta_c = 301.5$$

$$2EK\theta_b + 4EK\theta_c = -416.7$$

In Matrix Form we have:

$$\boxed{\begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} EK\theta_b \\ EK\theta_c \end{Bmatrix} = \begin{Bmatrix} 301.5 \\ -416.7 \end{Bmatrix}} \quad \text{—— (3)}$$

Example Slope Deflection Method

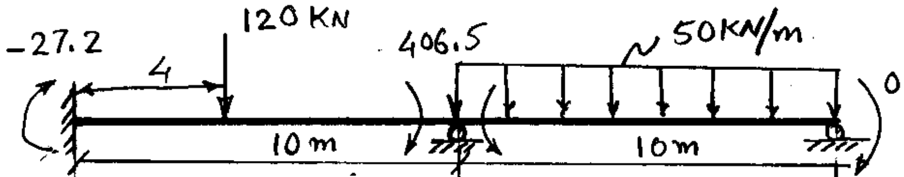
Solving the system of equations we have

$$\boxed{\begin{Bmatrix} EK\theta_b \\ EK\theta_c \end{Bmatrix} = \begin{Bmatrix} 72.8 \\ -140.6 \end{Bmatrix} \quad KN\text{-}M} \qquad \text{——— } (4)$$

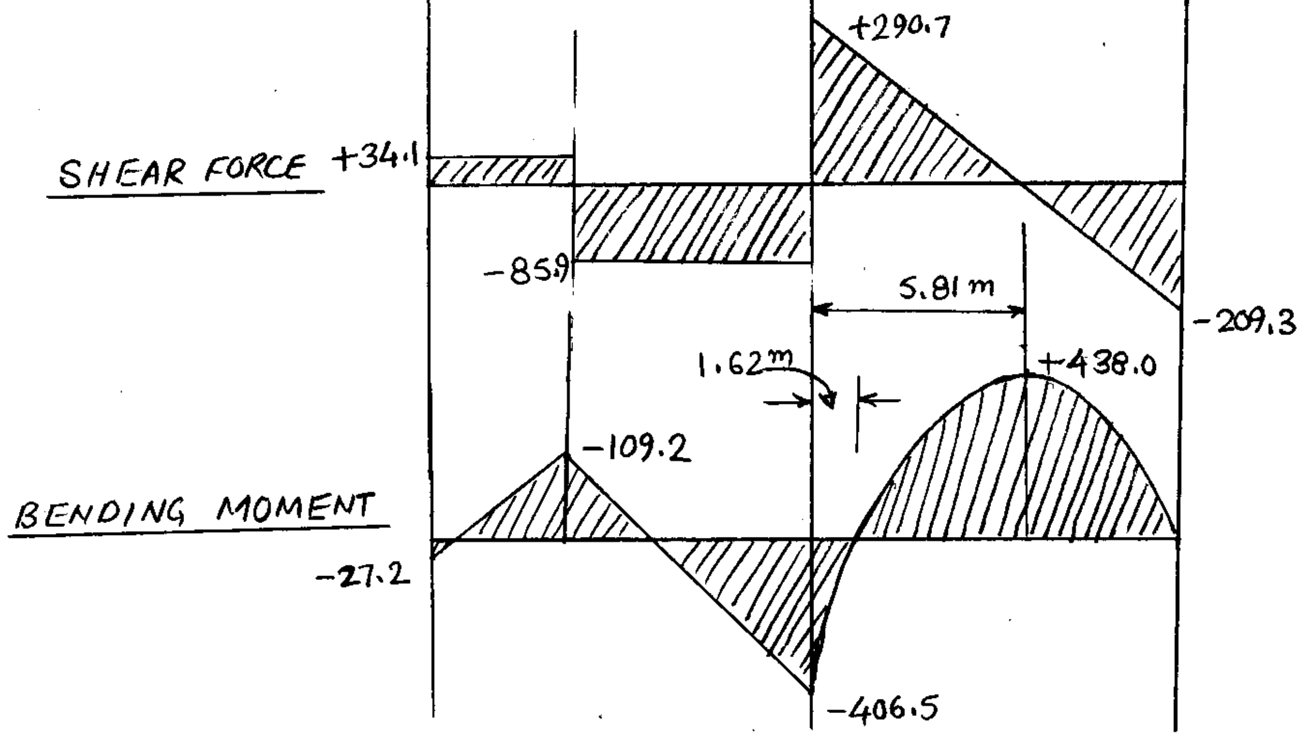
Substitute Displacements from Eq (4) into Moment Eqns (1) we have

$$\boxed{\begin{aligned}
M_{ab} &= 2EK\theta_b - 172.8 &&= 2(72.8) - 172.8 &&= -27.2 \text{ KN-m}\\
M_{ba} &= 2EK(2\theta_b) + 115.2 &&= 4(72.8) + 115.2 &&= +406.5\\
M_{bc} &= 2EK(2\theta_b + \theta_c) - 416.7 &&= 4(72.8) + 2(-140.6) - 416.7 &&= -406.5\\
M_{cb} &= 2EK(2\theta_c + \theta_b) + 416.7 &&= 4(-140.6) + 2(72.8) + 416.7 &&= 0 \text{ KN-m}
\end{aligned}}$$

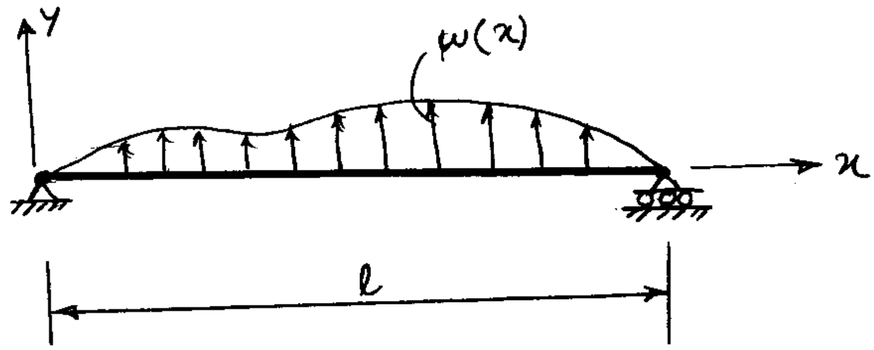
——— (5)



R Load	+72.0	+48.0	+250	+250
R Moment	-37.9	+37.9	+40.7	-40.7
R Total	+34.1	+85.9	+290.7	+209.3



CONJUGATE BEAM METHOD



For the Beam shown above following statics equation holds

$$\frac{d^2M}{dx^2} = \frac{dV}{dx} = w(x) \qquad \text{—— ①}$$

The Beam Deflection Problem is governed by the following differential equation.

$$\frac{d^2y}{dx^2} = \frac{d\theta}{dx} = \frac{M(x)}{EI} \qquad \text{—— ②}$$

Both eqns ① & ② are 2nd Order Linear Differential Equations

- The first integration of Eqn ① yields the beam shear V and the Second integration yields the Bending Moment.
- The first integration of Eqn ② yields the slope and the second integration yields the Beam Deflection.

Note that there is a correlation between eqns ① & ② such that

$$\left.\begin{array}{lcl} \frac{M}{EI} & \xleftrightarrow{\text{correlates}} & w \\ \theta & \longleftrightarrow & V \\ y & \longleftrightarrow & M \end{array}\right\} \qquad \text{—— ③}$$

CONJUGATE BEAM METHOD

From relations ③ we see that if a beam of same dimensions as the real beam is loaded by a fictitious loading $\frac{M(x)}{EI}$ then:

- The beam shear force would be equal to beam slope
- The beam bending moment would be equal to beam deflection.

Note that $\frac{M(x)}{EI}$ is equal to beam curvature $\frac{d^2y}{dx^2}$

This leads to the 2 Conjugate Beam Theorums:

1. The slope at any pt. of an actual beam subject to loading is given by the shear force (V) at the corresponding section of the conjugate beam subjected to the elastic load $\frac{M}{EI}$.

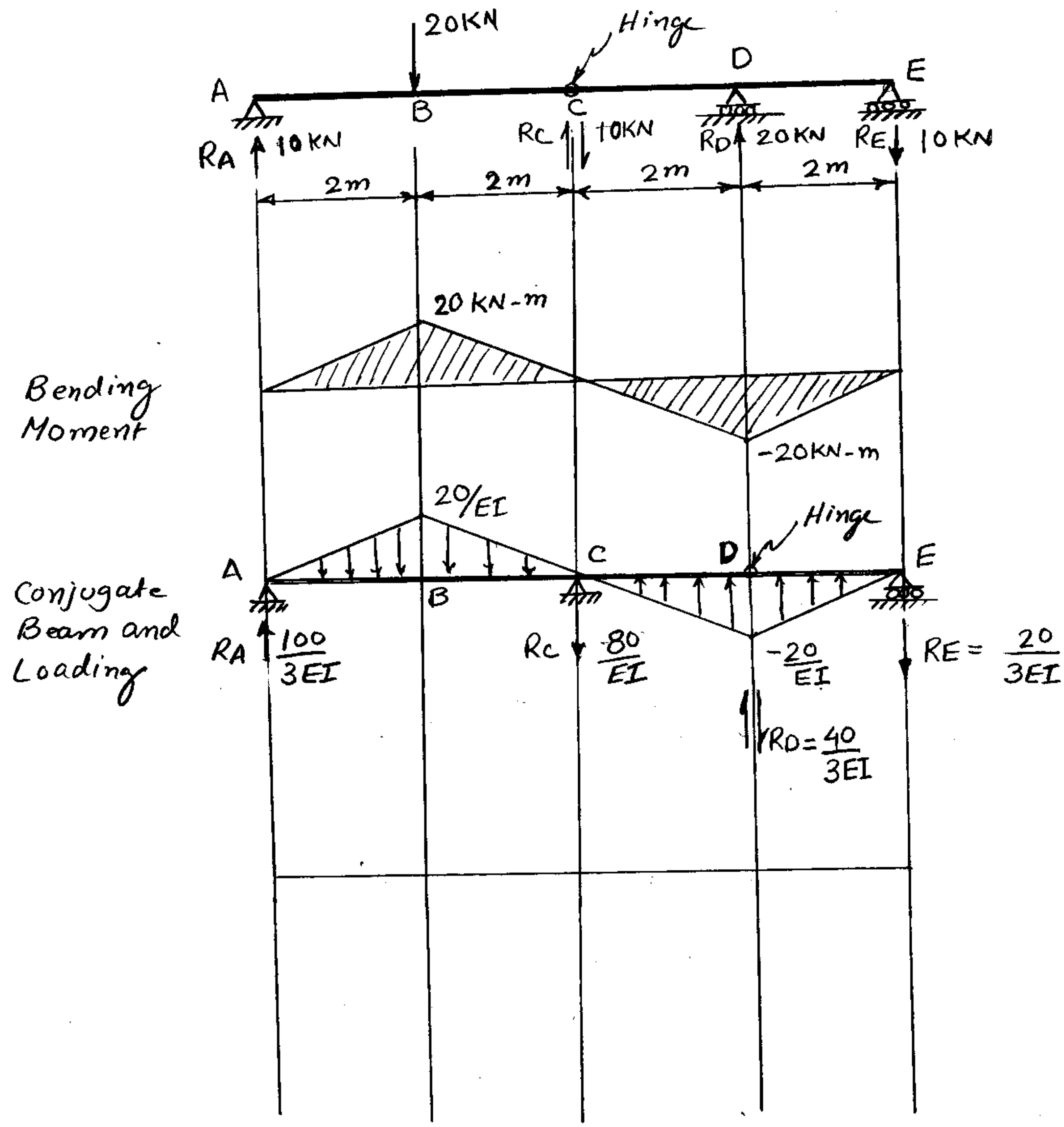
2. Similarly, deflection at any section is given by the corresponding Bending Moment at that section of the of the conjugate beam subjected to elastic load $\frac{M}{EI}$.

For the case of actual beam with support conditions other than simply supported, the support conditions may need to be modified such that the shear (V) and Bending Moment (M) in the Conjugate Beam conforms to the slopes (θ) and deflections (y) of the actual beam.

Support Conditions for the Conjugate Beam

	Actual Beam		Conjugate Beam
	Slope Deflection	⟷	Shear Bending Moment
BOUNDARY CONDITIONS	clamped	⟷	free
	pin	⟷	pin
	Free	⟷	Clamped
CONTINUITY CONDITIONS	Interior Support	⟷	Interior Hinge
	Interior Roller	⟷	Roller Transfer
	Interior Hinge	⟷	Interior Support

Example - Conjugate Beam Method



For the Beam shown above find the slopes and deflections at Pts B & C

Beam Properties $E = 20{,}000\ KN/cm^2$

$I = 5{,}000\ cm^4$

Example Conjugate Beam Method

Reaction in conjugate beam
@ Support E

Taking moments @ D

$$R_E \times 2 + \left(\frac{1}{2} \times \frac{20}{EI} \times 2\right) \times \frac{2}{3} = 0$$

$$2R_E + \frac{40}{3EI} = 0 \longrightarrow R_E = \frac{-20}{3EI} \downarrow$$

$$R_E + \frac{1}{2} \times \frac{20}{EI} \times 2 + R_D = 0$$

$$R_D = -\frac{20}{EI} + \frac{20}{3EI} \longrightarrow R_D = -\frac{40}{3EI} \downarrow$$

Taking moments about Pt C

$$R_A \times 4 - \left(\frac{1}{2} \times \frac{20}{EI} \times 4\right) \times 2 - \left(\frac{1}{2} \times \frac{20}{EI} \times 2\right) \times \frac{2}{3} \times 2 - \underbrace{\frac{40}{3EI}}_{R_D} \times 2 = 0$$

$$4R_A - \frac{80}{EI} - \frac{80}{3EI} - \frac{80}{3EI} = 0$$

$$\rightarrow R_A = \frac{400}{4 \times 3EI} = \frac{100}{3EI} \uparrow$$

$$R_A + R_C + R_E = 0$$

[Note: As Loads are self Cancelling]

$$\frac{100}{3EI} + R_C - \frac{20}{3EI} = 0 \longrightarrow R_C = \frac{-80}{3EI} \downarrow$$

Example Conjugate Beam Method

Now we can construct shear and Bending moment diagrams for the Conjugate Beam.

$$\text{Shear @ Pt } B = \frac{100}{3EI} - \frac{20 \times 2}{2\,EI} = \frac{100 - 60}{3EI}$$

$$= \frac{40}{3EI} \text{ clockwise}$$

→ $$\text{Slope @ Pt } B = \frac{40 \times 10^4}{3 \times 20{,}000 \times 5000} = 0.00133 \text{ radians.}$$

Deflection @ Pt B

$$\text{Bending moment @ Pt } B = \frac{100}{3EI} \times 2 - \frac{1}{2} \times \frac{20}{EI} \times 2 \times \frac{2}{3}$$

$$= \frac{200}{3EI} - \frac{40}{3EI}$$

$$= \frac{160}{3EI} = \frac{160 \times 10^4 \times 10^2}{3EI}$$

$$= 0.533 \text{ cm} \downarrow$$

Rotations @ Hinge C

$$\text{Shear Left of } C = \frac{100}{3EI} - \frac{40}{EI} = \frac{100 - 120}{3EI}$$

$$= \frac{-20}{3EI} = \frac{-20 \times 10^4}{3 \times 20{,}000 \times 5000} = -0.00066 \text{ radians Anticlockwise}$$

$$\text{Shear Right of } C = \frac{-20}{3EI} - \frac{80}{3EI}$$

$$= -\frac{100}{3EI} = \frac{-100 \times 10^4}{3 \times 20000 \times 5000} = -0.0033 \text{ rads Anticlockwise}$$

$$\text{Deflection @ } C = \text{BM @ } C$$

$$= \frac{100}{3EI} \times 4 - \frac{1}{2} \times \frac{20}{EI} \times 4 \times 2$$

$$= \frac{160}{3EI} = \frac{160 \times 10^4 \times 10^2}{3 \times 20000 \times 5000} = 0.533 \text{ cm} \downarrow$$