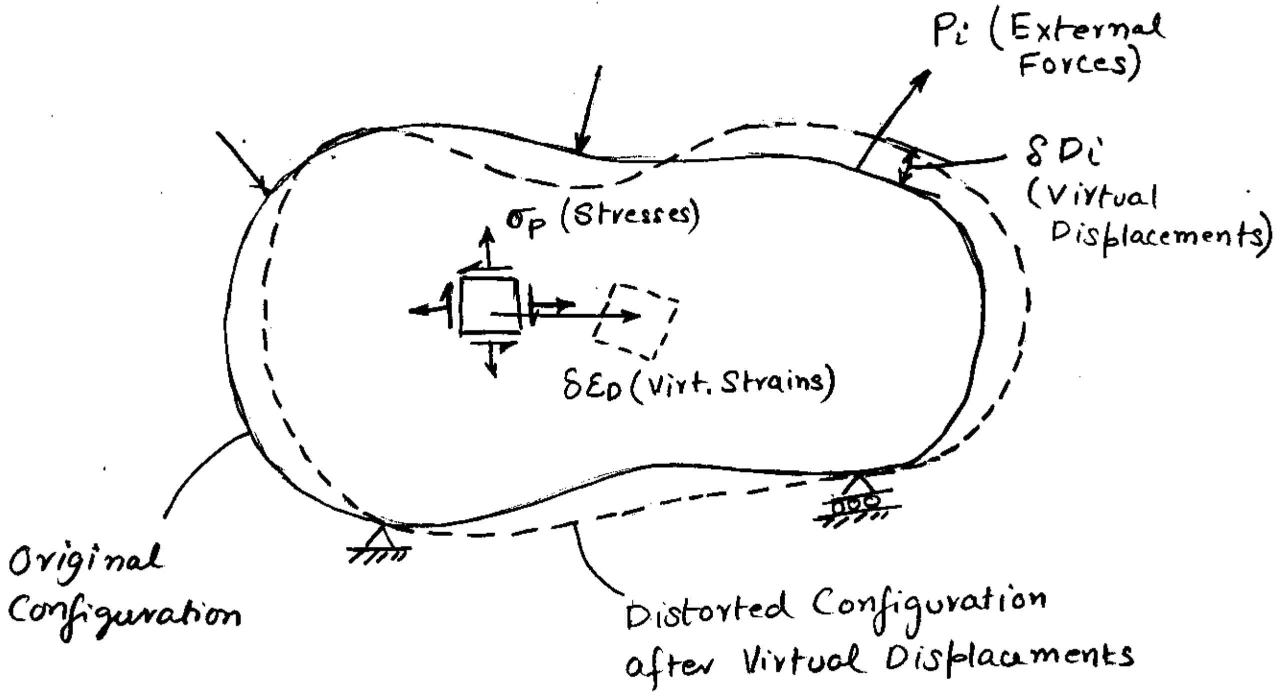
VIRTUAL WORK PRINCIPLES



Consider the Deformable Body shown above which is in equilibrium under a system of forces "P".

The Body is then subjected to a virtual distortion which results in virtual displacements "δD_i" at points of load application.

Note: The imposed virtual distortion field satisfies the support boundary conditions.

σ_P = Stresses in the Body due to the applied Real Loads P_i

$\delta \varepsilon_D$ = Virtual Strains generated in the Body due to imposed Virtual Displacement Field.

δW_e = Virtual Work done by the external forces under virtual displacement field

$= \sum P_i \, \delta D_i$

δW_d = Strain Energy stored in the body as the actual stresses σ_P undergo virtual strains $\delta \varepsilon_D$

VIRTUAL WORK PRINCIPLES

The virtual work done by a small element of the body is denoted by $d(\delta W_d)$

$$d(\delta W_d) = (\sigma_P \cdot dA)(\delta \varepsilon_D \cdot dl)$$

The Total Strain Energy generated in the body δW_D is given by:

$$\delta W_d = \int_{Vol} (\sigma_P \cdot dA)(\delta \varepsilon_D \cdot dl)$$

$$\boxed{\delta W_d = \int_{Vol} \sigma_P \cdot \delta \varepsilon_D \, dvol}$$

Equating the External Virtual Work to the Virtual Strain Energy we have:

$$\boxed{\delta W_e = \delta W_d}$$

Actual Forces/Stresses

$$\sum P_i (\delta D_i) = \int_{Vol} \sigma_P (\delta \varepsilon_D) \, dvol$$

Virtual Displ. / Strains

Principle of Virtual Work Using Virtual Displacements

Above equation is the "Principle of Virtual Displacements" which essentially states that If a deformable body is in equilibrium under a set of forces P and is subjected to virtual displaments/distortion that satisfies the essential boundary conditions, the virtual work done by the forces P is equal to the internal virtual work done by actual stresses σ_P

VIRTUAL WORK PRINCIPLES

By Selecting a proper set of δD virtual displacements the desired P force can be isolated on the left hand side of the virtual work expression and hence determined.

VIRTUAL WORK METHOD USING VIRTUAL FORCES

The Principle of Virtual Work in conjunction with virtual forces can be used to find the desired displacements as follows:

Virtual Force / Stresses

$$\sum (\delta P_i)\, D_i = \int_{Vol} (\delta \sigma_P)\, \varepsilon_D \; d\,Vol$$

Actual Forces / Strains

Principle of Virtual Work Using Virtual Forces.

By selecting a proper set of δP Virtual Forces the desired displacements / deflections at desired locations can be determined.

For example the "Unit Load / Dummy Load Method" is based on applying a unit virtual load at the point where deflection is desired to be determined

$$\delta P_i\, D_i = D_i = \int_{Vol} (\delta \sigma_P)\, \varepsilon_D \; d\,Vol$$

Unit load

Desired Displacement

Virtual Stresses due to unit load

Actual Strains

UNIT LOAD METHOD

PRINCIPLE OF MINIMUM POTENTIAL ENERGY

"The Principle of Minimum Potential Energy states that for a system to be in equilibrium, the variation in the Total Potential Energy must vanish for any virtual deformation." In other words the Potential Energy is "Stationary" with respect to variations in the displacements.

The Principle of Minimum Potential Energy can be derived from the Principle of Virtual Work

$$\delta W_e = \delta W_d$$

δW_e = External Virtual Work

δW_d = Internal Virtual Work due to imposed virtual distortion.

$\delta W_d = \delta U$ = Virtual Change in the strain energy

Hence we have

$$\delta U - \delta W_e = 0$$

or

$$\boxed{\delta(U - W_e) = \delta(\Pi_p) = 0}$$ — Principle of Minimum Potential Energy.

The quantity $\Pi_p = U - W_e$ is called the Potential Energy of the system.

Another way of stating the "Principle of Minimum Potential Energy" may also be stated as "Amongst all possible sets of deformations, that which ensures that all the equilibrium conditions are fulfilled will lead to minimization of total Potential Energy Π_p i.e

$$\boxed{\delta \Pi_p = \sum \frac{\partial \Pi_p}{\partial D_i} \delta D_i = 0}$$

PRINCIPLE OF MINIMUM POTENTIAL ENERGY

The δD_i are independent first variations of displacements. Since δD_i are arbitrary the variation in Total Potential Energy Π_p for equilibrium can only be zero if

$$\boxed{\frac{\partial \Pi_p}{\partial D_i} = 0}$$

For all displacement degrees of freedom D_i

— Principle of Minimum Potential Energy (Equilibrium/Stationarity Condition)

The Principle of Minimum Potential Energy is valid for linear and nonlinear systems.

CASTIGLIANO'S FIRST THEOREM

When the change in Total Potential Energy (Π_p) results from a virtual change in the ith Displacement Degree of freedom D_i, we have from the Principle of Minimum Potential Energy:

$$\delta \Pi_p = \delta U - \delta W_e = \frac{\partial U}{\partial D_i}\,\delta D_i - P_i\,\delta D_i = 0$$

δU = Virtual change in Strain Energy

δW_e = External Virtual Work.

δD_i = Virtual Displacement corresponding to Force P_i

$$\left(\frac{\partial U}{\partial D_i} - P_i\right)\delta D_i = 0$$

As δD_i is arbitrary virtual displacement

$$\boxed{\frac{\partial U}{\partial D_i} = P_i}$$

— Castigliano's 1st Theorem

CASTIGLIANO'S 1ST THEOREM

Castigliano's 1st Theorum states that Partial Derivative of the Strain Energy with respect to any displacement degree of freedom D_i is equal to the force P_i corresponding to the displacement.

$$\boxed{\frac{\partial U}{\partial D_i} = P_i}$$

Castigliano's 1st Theorum is applicable to linear and nonlinear elastic structures.

CASTIGLIANO'S 2ND THEOREM

If the change in Total Potential Energy (π_P) results from a virtual change in the ith Force Quantity P_i, the Principle of Minimum Potential Energy gives:

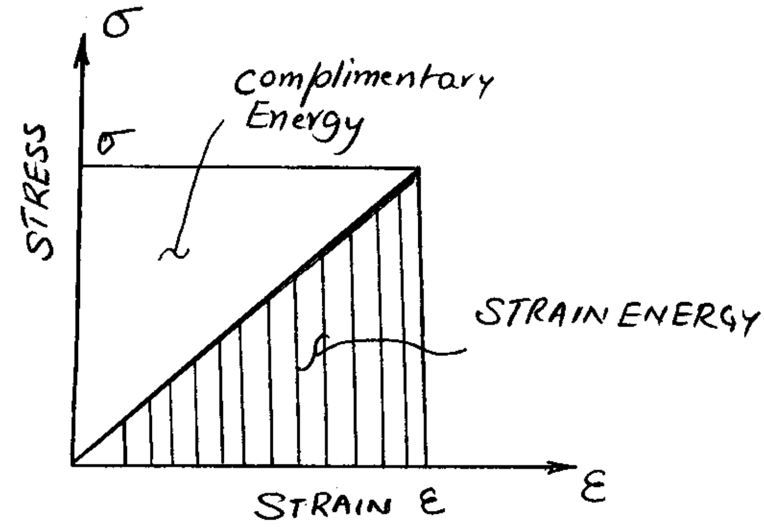
$$\delta \pi_P = \delta U - \delta W_e = \frac{\partial U}{\partial P_i} \delta P_i - \delta P_i D_i = 0$$

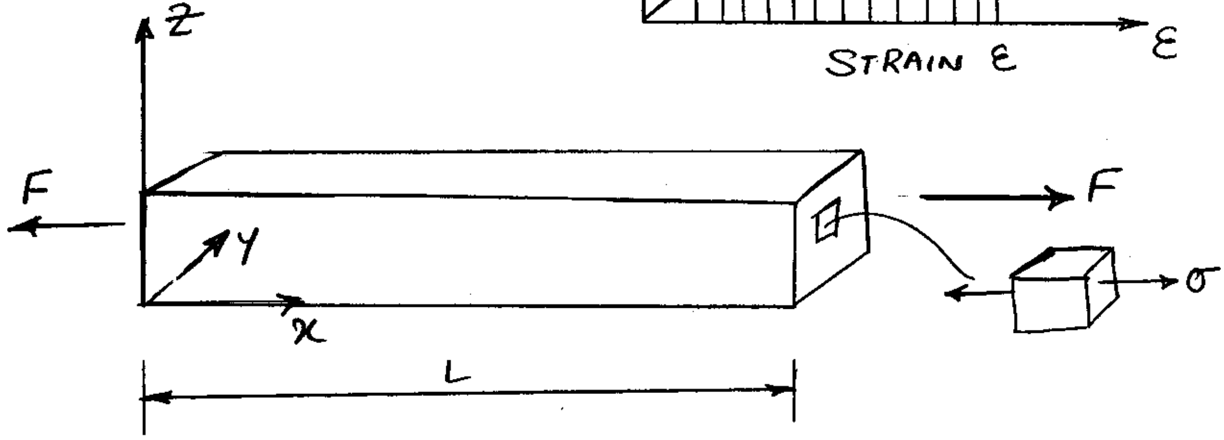
$$\Rightarrow \left(\frac{\partial U}{\partial P_i} - D_i \right) \delta P_i = 0 \qquad \text{since } \delta P_i \neq 0$$

$$\Rightarrow \boxed{\frac{\partial U}{\partial P_i} = D_i}$$ — Castigliano's 2nd Theorem.

Castigliano's 2nd Theorum states that Partial Derivative of the Strain Energy with respect to any force quantity P_i is equal to the displacement D_i corresponding to that Force

STRAIN ENERGY OF A LINEAR ELASTIC MEMBER





STRAIN ENERGY OF A BAR UNDER AXIAL FORCE

The Strain Energy Density at a point under axial stress $= u = \frac{1}{2}\sigma . \varepsilon$

Total Strain Energy in the bar $= U = \int u \, dv$

$$U = \frac{1}{2}\int \sigma . \varepsilon \, dv$$

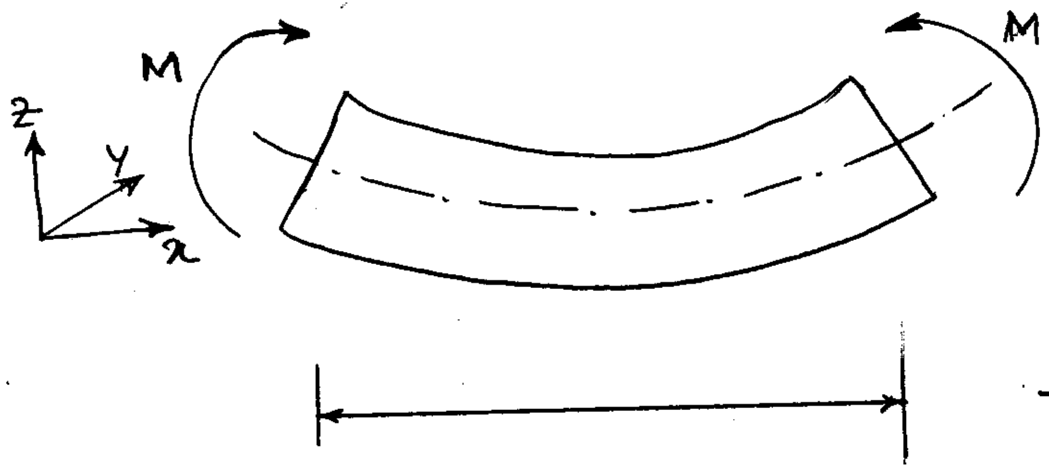
$$U = \frac{1}{2}\int_{Vol} \sigma . \frac{\sigma}{E} \, dx\,dy\,dz$$

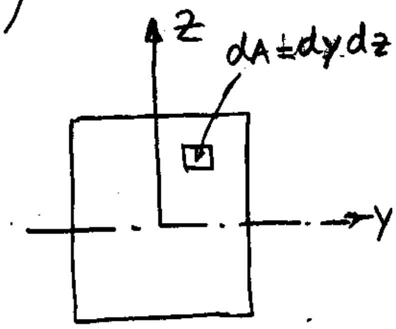
$$\sigma = \frac{F}{A}$$

$$U = \frac{1}{2}\int \frac{F^2}{A^2E} \, dx\,dy\,dz = \frac{F^2}{2AE}\int_{Vol} dx\,dy\,dz$$

$$U = \frac{1}{2}\frac{F^2}{A^2E}(A . L) \Rightarrow \boxed{U = \frac{1}{2}\frac{F^2L}{AE}}$$

STRAIN ENERGY OF BENDING





Strain Energy Density $= u_0 = \frac{1}{2}\sigma . \varepsilon$

$$\sigma = \frac{M}{I} . z \quad , \quad \varepsilon = \frac{\sigma}{E} = \frac{M}{EI} . z$$

$$u = \frac{1}{2}\left(\frac{M}{I} . z\right)\left(\frac{M}{EI} . z\right)$$

$$u = \frac{1}{2}\frac{M^2}{EI^2} z^2$$

$$U = \int u \, dx\, dy\, dz$$

$$= \frac{1}{2E}\int \frac{M^2}{I^2} z^2 \, dx\, dy\, dz$$

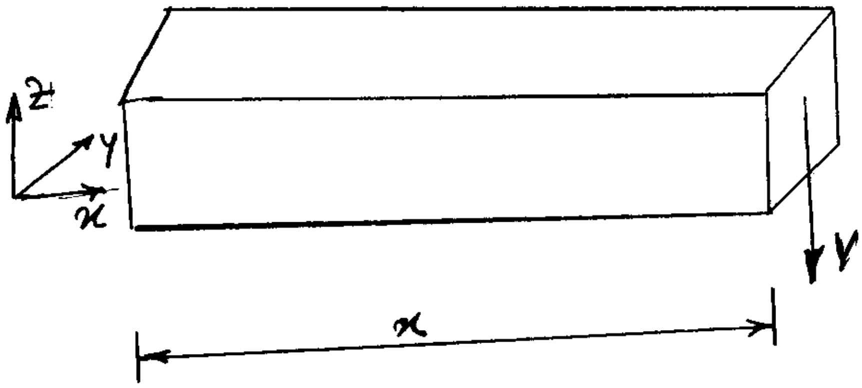
$$= \frac{1}{2E}\int \frac{M^2}{I^2} dx \int z^2 \, dy\, dz$$

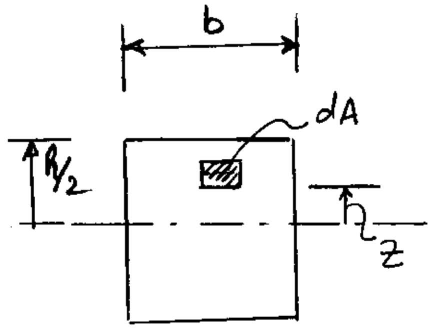
$$= \frac{1}{2E}\int_l \frac{M^2}{I^2} dx \underbrace{\int_A z^2 . dA}_{I}$$

$$= \frac{1}{2E}\int \frac{M^2 dx}{I^2} . I$$

$$\boxed{U = \frac{1}{2EI}\int M^2 dx}$$

STRAIN ENERGY OF SHEAR





Strain Energy Density $= u = \frac{1}{2}\tau_{xz}\gamma_{xz}$

$$\tau_{xz} = \frac{VQ}{Ib} \quad , \quad \gamma_{xz} = \frac{\tau_{xz}}{G} = \frac{VQ}{IbG}$$

$$Q = \int_{z}^{h/2} z . dA$$

$$u = \frac{1}{2}\frac{VQ}{Ib} . \frac{VQ}{IbG} = \frac{1}{2}\frac{V^2Q^2}{I^2b^2G}$$

$$\text{STRAIN ENERGY} = \int u\, dv = \frac{1}{2G}\int \frac{V^2Q^2}{I^2b^2G} . dxdydz$$

$$= \frac{1}{2G}\int_{l} V^2 dx . \int \frac{Q^2}{I^2b^2} . dydz$$

Now $\int_A \frac{Q^2}{I^2b^2} dydz = \frac{1}{I^2}\int_A \left(\frac{Q}{b}\right)^2 dA = \frac{1}{A_s}$

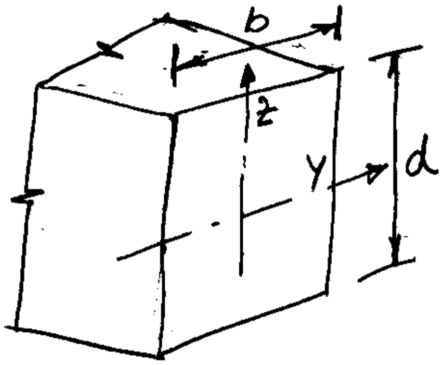
A_s = Effective Shear Area

$$\Rightarrow \boxed{U = \frac{1}{2G}\int \frac{V^2}{A_s} dx}$$

STRAIN ENERGY OF SHEAR.

STRAIN ENERGY OF SHEAR

Effective Shear Area (A_s) of a rectangular Section.



$$\frac{1}{A_s} = \frac{1}{I^2}\int\left(\frac{Q}{b}\right)^2 dA$$

$$Q_{\substack{\text{rectangular}\\ \text{section}}} = \frac{b}{2}\left(\frac{d^2}{4}-z^2\right) \quad , \quad I = \frac{bd^3}{12}$$

$$\frac{1}{A_s} = \frac{1}{\left(\frac{bd^3}{12}\right)^2}\cdot\int_A\left[\frac{b}{2}\left(\frac{d^2}{4}-z^2\right)\cdot\frac{1}{b}\right]^2 b.\,dz$$

$$= \frac{144}{b^2d^6}\cdot\frac{b}{4}\int_{-d/2}^{d/2}\left(\frac{d^2}{4}-z^2\right)^2 dz$$

$$\frac{1}{A_s} = \frac{36d^5}{bd^6}\cdot\frac{16}{480} = \frac{12}{bd}$$

$$\Rightarrow \boxed{A_{s\,\substack{\text{Rectangular}\\ \text{Section}}} = \frac{bd}{12}}$$

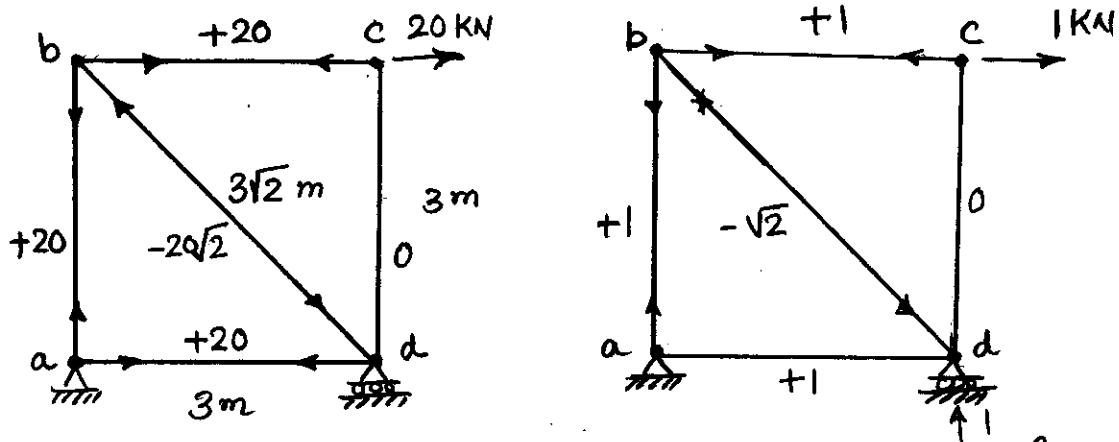
$$\Rightarrow U = \frac{1}{2G}\int\frac{V^2}{A_s}dx$$

$$U = \frac{1}{2G}\cdot\frac{12}{bd}\int V^2 dx$$

$$\boxed{U = \frac{6}{Gbd}\int V^2 dx}$$

Strain Energy of Shear for a rectangular Section.

Example Problem



For the Truss shown above find the horizontal deflection at Pt C. Area of Truss Members = $1000\ mm^2$ and E = 200 GPa. Use the unit load method and method of virtual work.

Virtual Work Expression using Virtual Forces is

$$\sum (\delta P_i)\, D_i = \int_{Vol} \delta\sigma_P\ \varepsilon_D\ dVol$$

$$= \sum \int_{l} (\delta\sigma_P \cdot dA)\ \varepsilon_D\ dx$$

$$= \sum \int_{l} \delta F_P\ \frac{F_D}{EA}\ dx$$

$$\boxed{\sum (\delta P_i)\, D_i = \sum_{j=1}^{m} \left(\delta F_P \cdot \frac{F_D\, l}{EA}\right)_j}$$

Virtual Work Expression for Trusses

Example Problem

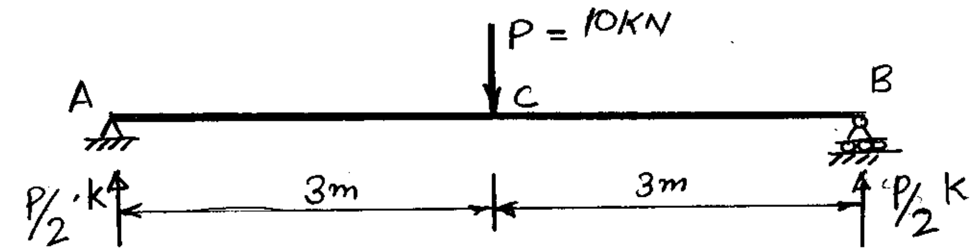
Member	l (m)	Member Force F_D (KN)	Virtual Member Force δF_P (KN)	$\delta F_P . F_D . l$ $(KN)^2 . m$
ab	3	+20	+1	60
bc	3	+20	+1	60
cd	3	0	0	0
ad	3	+20	+1	60
bd	$3\sqrt{2}$	$-20\sqrt{2}$	$-\sqrt{2}$	$+120\sqrt{2}$
		$\sum \delta F_P . F_D . l$		349.706 $KN^2 . m$

$$\Delta_{c\ horiz} = \frac{1}{EA} \sum_{j=1}^{m} (\delta F_P . F_D . l)$$

$$= \frac{349.706 \times 1000}{1000 \times 200} = 1.748 \text{ mm}$$

in Direction of Unit Load

Example Problem



$E = 200\ GPa$

$I = 500\ mm^4$

Find Deflection at the Point of application of the load using Castigliano's 2nd Theorem.

$$\frac{\partial U}{\partial P_i} = D_i$$

$$U = \frac{1}{2EI}\int M^2 dx \quad , \quad M = \frac{5}{2} x$$

$$U = \left[\frac{1}{2EI}\int_0^{l/2}\left(\frac{P}{2}x\right)^2 dx\right] \times 2$$

$$= \frac{P^2}{4EI}\int_0^{l/2} x^2 dx = \frac{P}{4EI}\left|\frac{x^3}{3}\right|_0^{l/2} = \frac{P^2}{4EI}\left(\frac{l^3}{24}\right)$$

$$U = \frac{P^2 L^3}{96 EI}$$

$$\frac{\partial U}{\partial P_i} = \frac{2PL^3}{96EI} = \frac{PL^3}{48EI} = D_c$$

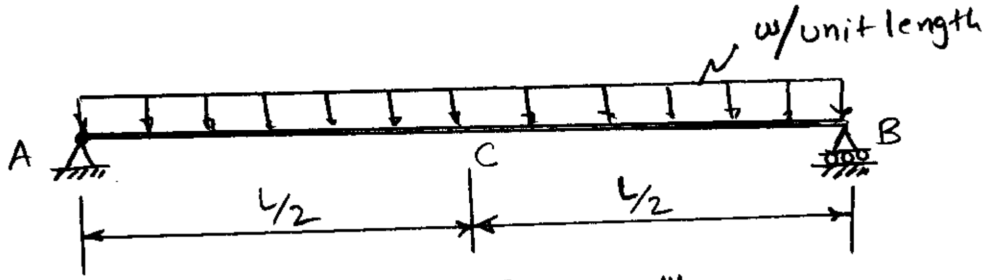
$\Rightarrow$ Deflection @ mid span $= \dfrac{PL^3}{48EI}$

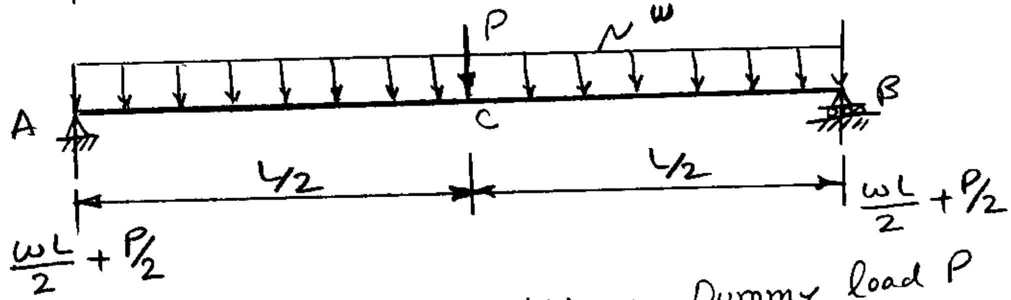
For $P = 10KN$, $E = 200\ GPa$ and $I = 500\ mm^4$

$$\text{Deflection} = \frac{(10)(6)\ 1000^4}{48 \times 200 \times 10^3 \times 500} = 0.125\ mm \downarrow$$

Example

For the Beam shown below find the central deflection due to uniformly distributed load w using Castigliano's 2nd Theorem





To find deflection @ C we apply a Dummy load P at Pt C

$$U = \frac{1}{2EI}\int M^2\, dx$$

$$D_c = \frac{\partial U}{\partial P} = \frac{1}{2EI}\int_{l} 2M\left(\frac{\partial M}{\partial P}\right)dx$$

$$= \frac{1}{EI}\int_0^{l} M\left(\frac{\partial M}{\partial P}\right)dx.$$

$$M = \left(\frac{wL}{2} + P/2\right)x - \frac{wx^2}{2}, \quad 0 \leq x \leq L/2$$

$$\frac{\partial M}{\partial P} = \frac{x}{2}$$

$$D_c = \frac{2}{EI}\int_0^{l/2}\left[\left(\frac{wL}{2} + P/2\right)x - \frac{wx^2}{2}\right]\cdot\frac{x}{2}\, dx$$

$$= \frac{2}{EI}\int_0^{l/2}\left[\frac{wlx^2}{4} + \frac{Px^2}{4} - \frac{wx^3}{4}\right]dx$$

Example

$$D_c = \frac{2}{4EI}\int_0^{l/2}\left[(wl+P)x^2 - wx^3\right]dx$$

$$= \frac{1}{2EI}\left|(wl+P)\frac{x^3}{3} - \frac{wx^4}{4}\right|_0^{l/2}$$

as $P = 0$

$$D_c = \frac{1}{2EI}\left[wl\left(\frac{l^3}{24}\right) - \frac{wl^4}{64}\right]$$

$$= \frac{wl^4}{2EI}\left[\frac{1}{24} - \frac{1}{64}\right]$$

$$= \frac{wl^4}{2EI}\left(\frac{8-3}{192}\right)$$

$$\boxed{D_c = \frac{5wl^4}{384EI}}$$

Answer