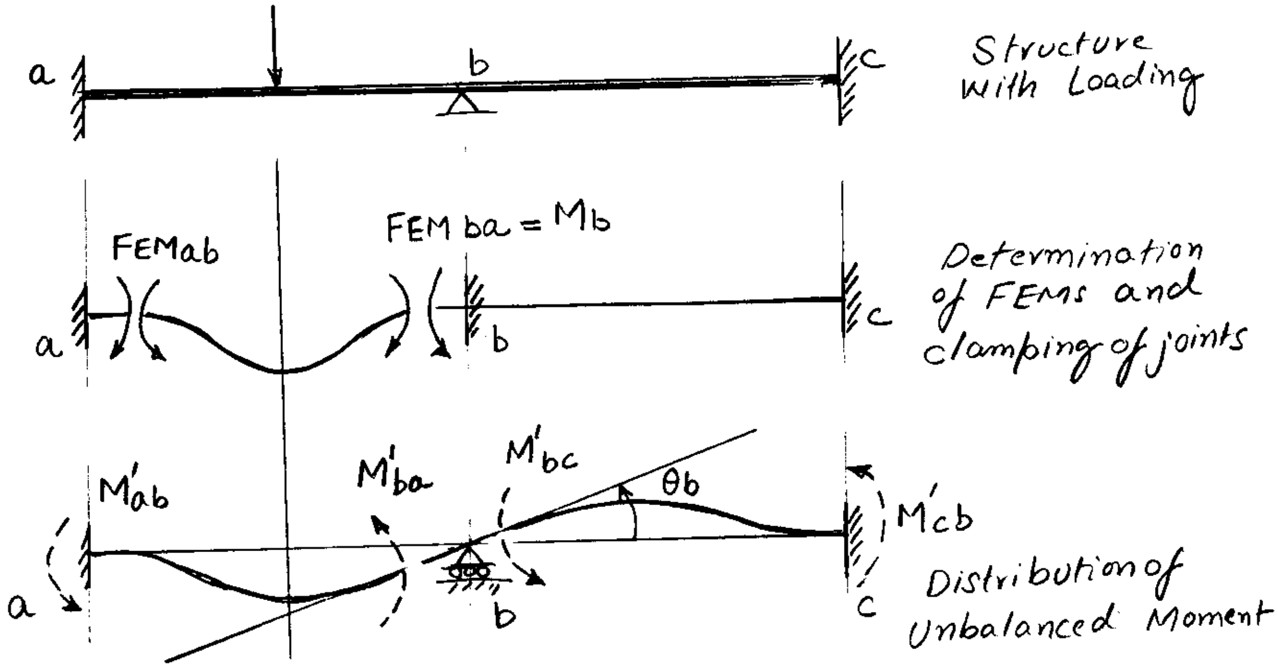
MOMENT DISTRIBUTION METHOD

- Moment Distribution Method is an iterative solution method for beam and frame structures.
- The method was developed by Prof. Hardy Cross who taught it to his students at University of Illinois in 1920's.
- Hardy Cross formally presented the method in 1930 in published form.
- Moment Distribution remained the dominant method of analyzing structures well into 1950's.
- The method works by starting from a condition when all the joints of the structure are clamped. Then the joints are released and unbalanced moments are distributed throughout the structure and the joints are again clamped.
- The distribution of moments causes the joints to again be out of balance. These out of balance moment are again redistributed throughout the structure.
- The above mentioned process of clamping the joints, redistributing the out of balance moments is carried out repeatedly till the out of balance moments are negligible and all the joints are in balanced state.

Moment Distribution Method



Consider the beam shown above that is clamped at ends "a" and "c". Rotation is possible at Pt "b"

In the First step all the joints are locked. This gives rise to development of Fixed End Moments at the member ends.

The unbalanced moment at end "b" is FEMba = Mb

The joint "b" is now unlocked and the moment Mb is allowed to be distributed between members ba and bc. The rotation at joint "b" θ_b will occur till the joint is in equilibrium i.e. till:

$$M'_{ba} + M'_{bc} = -M'_b \qquad (1)$$

Recalling the Slope-Deflection Equation

$$M'_{nf} = 2EK_{nf}\,(2\theta_n + \theta_f) \qquad (2)$$

since $\theta_a = \theta_c = 0$

$$\left.\begin{aligned} M'_{ba} &= 4EK_{ba}\,\theta_b \\ M'_{bc} &= 4EK_{bc}\,\theta_b \end{aligned}\right\} \qquad (3)$$

Substitution of Equations ③ in Equation ① we have

$$4K_{ba}\theta_b + 4K_{bc}\theta_b = -M_b'$$

$$\Rightarrow \quad \theta_b = \frac{-M_b'}{4E(K_{ba}+K_{bc})} \quad \text{———} ④$$

Substituting Equation ④ into Equations ③ we have

$$M_{ba}' = -\left(\frac{K_{ba}}{K_{ba}+K_{bc}}\right) M_b' \quad \text{———} ⑤$$

$$M_{bc}' = -\left(\frac{K_{bc}}{K_{ba}+K_{bc}}\right) M_b' \quad \text{———} ⑥$$

Or in General,

$$\boxed{M_{bi}' = -\left(\frac{K_{bi}}{\sum_j K_{bj}}\right) \cdot M_b'} \quad \text{———} ⑦$$

$$\boxed{M_{bi}' = -D_{bi} \cdot M_b'}$$

$$\boxed{D_{bi} = \text{Distribution Factor} = \frac{K_{bi}}{\sum_j K_{bj}}} \quad \text{———} ⑧$$

D_{bi} (Distribution Factor) relates the stiffness of member bi to the sum of stiffnesses of all the members framing into joint "b"

Equation ⑦ also shows that the direction of distributed moments M_{bi} would be opposite to the unbalanced moment at joint "b"

Moment Distribution Method

As joint "b" rotates and moments develop at ends "b" of the members framing into joint "b", moments also develop at joints "a" and "c" that are locked.

From the Slope-Deflection equations we have:

$$\left.\begin{aligned} M'_{ab} &= 2EK_{ab}(\theta_b) \\ M'_{cb} &= 2EK_{cb}(\theta_b) \end{aligned}\right\} \text{———— (9)}$$

Let us Recall Eqns (3):

$$\left.\begin{aligned} M'_{ba} &= 4EK_{ba}\,\theta_b \\ M'_{bc} &= 4EK_{bc}\,\theta_b \end{aligned}\right\} \text{———— (3)}$$

Comparing Equations (9) & (3) we conclude that:

$$\boxed{\begin{aligned} M'_{ab} &= \frac{1}{2} M'_{ba} \\ M'_{cb} &= \frac{1}{2} M'_{bc} \end{aligned}} \text{———— (10)}$$

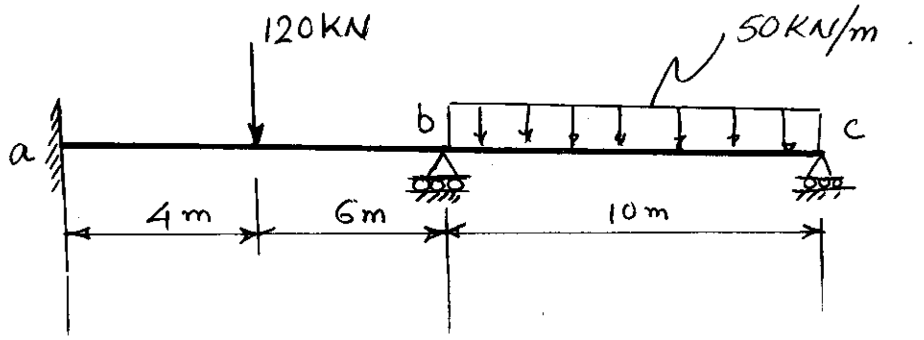
or in general Notation :

$$\boxed{\begin{aligned} M'_{ib} &= \frac{1}{2} M'_{bi} \\ \text{or} \quad M'_{ib} &= C_{bi} \cdot M'_{bi} \\ \text{Where } C'_{bi} &= \tfrac{1}{2} = \text{Carry-Over Factor} \end{aligned}} \text{———— (11)}$$

C'_{bi} (Carry Over Factor) is the factor that must be applied to M'_{bi} to determine M'_{ib}. M'_{ib} being the moment that gets "carried-over" to end i of member bi as joint b rotates.

Moment Distribution Method

Numerical Example



Stiffnesses & Relative Stiffnesses

$$K_{ab} = K_{ba} = \left(\frac{I}{L}\right)_{ab} = \frac{I}{10} = K$$

$$K_{bc} = K_{cb} = \left(\frac{I}{L}\right)_{bc} = \frac{I}{10} = K.$$

Distribution Factors

$$D_{bi} = \frac{K_{bi}}{\sum_j K_{bj}}$$

At Joint b:

$$D_{ba} = \frac{K_{ba}}{K_{ba} + K_{bc}} = \frac{K}{K+K} = 0.5$$

$$D_{bc} = \frac{K_{bc}}{K_{ba} + K_{bc}} = \frac{K}{K+K} = 0.5$$

At Joint c:

$$D_{cb} = \frac{K_{cb}}{K_{cb}} = \frac{K}{K} = 1.0$$

At Joint a:

D_{ab} = Undefined as it is a fixed end.

Moment Distribution Method

Example

Fixed End Moments :-

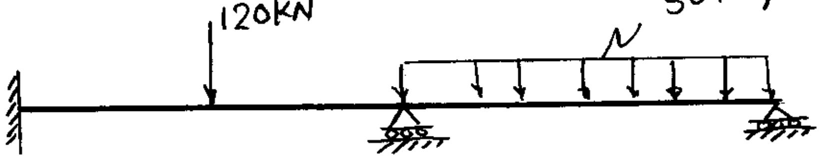
$$FEM_{ab} = -\frac{Pab^2}{\ell^2} = -\frac{120 \times 4 \times 6^2}{10^2} = -172.8 \text{ KN-m.}$$

$$FEM_{ba} = \frac{Pa^2b}{\ell^2} = \frac{120 \times 4^2 \times 6}{10^2} = +115.2 \text{ KN-m}$$

$$FEM_{bc} = -\frac{\omega \ell^2}{12} = \frac{-50 \times 10^2}{12} = -416.67 \text{ KN-m}$$

$$FEM_{cb} = \qquad = +416.67 \text{ KN-m}$$

Moment Distribution



Dist. Factor	—	0.5	0.5	1.0
Carry Over	←	0.5	0.5	0.5
FEM	−172.8	+115.2	−416.7	+416.7
	+75.4 ←	+150.75	+150.75 →	+75.4
			−246.05 ←	−492.1
	+61.5 ←	+123.03	+123.03 →	+61.5
			−30.75 ←	−61.5
	+7.7 ←	+15.4	+15.4 →	+7.7
			−3.85 ←	−7.7
	+0.97 ←	+1.93	+1.93 →	+0.97
			−0.49 ←	−0.97
	+0.12 ←	+0.24	+0.24	
FINAL MOMENTS	−27.1	+406.6	−406.5	0

MOMENT DISTRIBUTION METHOD

SUPPORT SETTLEMENT

The Moment Distribution Method is an iterative solution procedure for the following Slope-Deflection Equation.

$$M_{nf} = 2EK_{nf}(2\theta_n + \theta_f) + FEM \quad \text{——— ①}$$

The Fixed End Moments are computed and considered first: subsequently, the effect of joint rotations is considered in an iterative manner till all the joints are in a balanced condition and the carry over moments become negligible.

When the support-settlements are present, the Slope-Deflection Equations become:

$$M_{nf} = 2EK_{nf}(2\theta_n + \theta_f) - \frac{6EK_{nf}\Delta_{nf}}{l} + FEM_{nf} \quad \text{— ②}$$

* To make the support settlement problem amenable to moment distribution method the fixed end moments due to support settlements are considered along with the fixed end moments due to imposed loads. The moment distribution method can then then be applied as usual.

Thus if:

$$FEM^{S}_{nf} = \text{Fixed End Moments Due to Settlement}$$

$$= \frac{-6EK_{nf}\Delta_{nf}}{l}$$

and

$$FEM_{nf} = \text{Fixed End Moments due to imposed loads}$$

Then

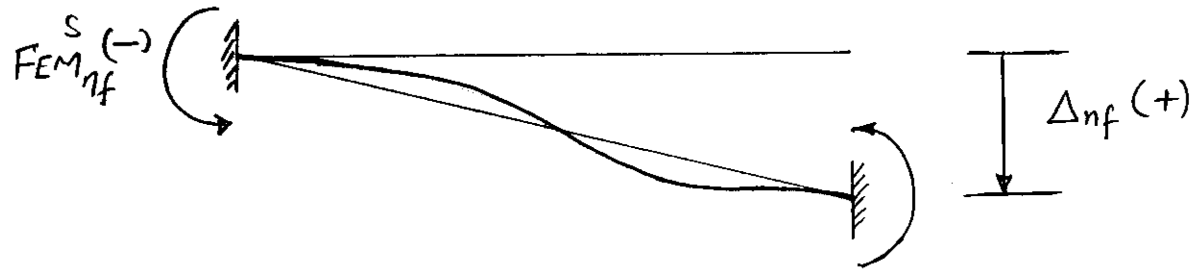
$$\boxed{FEM^{T}_{nf} = FEM^{S}_{nf} + FEM_{nf}} \quad \text{——— ③}$$

The Slope-Deflection Equation becomes:

$$\boxed{M_{nf} = 2EK_{nf}(2\theta_n + \theta_f) + FEM^{T}_{nf}} \quad \text{——— ④}$$

Moment Distribution Method

Support Settlement



FEM_{nf} = −ive if settlement downwards

FEM_{nf} = +ive if settlement upwards.

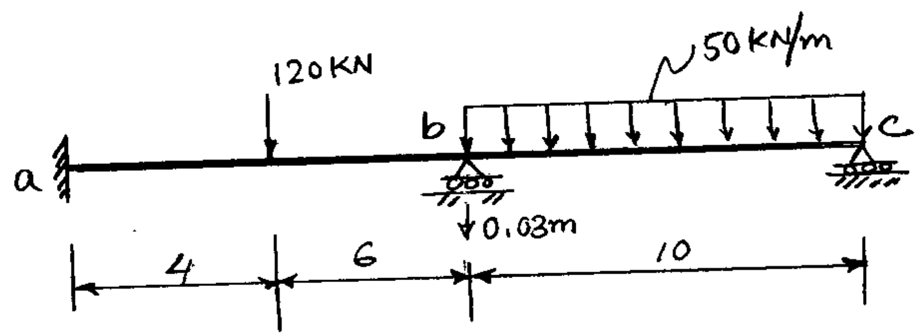
$$\boxed{FEM_{nf} = FEM_{fn} = -\frac{6EK_{nf}\Delta_{nf}}{l}} \qquad (5)$$

Moment Distribution Method

Example - Support Settlement

Solve the following problem if in addition to the applied loading the support at pt "b" also settles downward by 0.03 m

Given $E = 200$ GPa , $I = 2000 \times 10^{-6}\ m^4$



Relative Stiffnesses

Same as previously computed

Distribution Factors

Same as previously computed

$D_{ba} = 0.5$

$D_{bc} = 0.5$

$D_{cb} = 1.0$

Fixed End Moments Due to Loads

$FEM_{ab} = -172.8$ KN-m

$FEM_{ba} = +115.2$ KN-m

$FEM_{bc} = -416.7$ KN-m

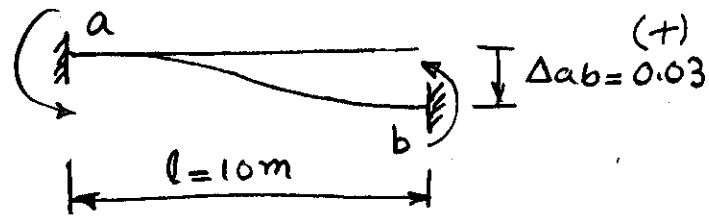
$FEM_{cb} = +416.7$ KN-m

Moment Distribution Method

Example - Support Settlement

Fixed End Moments Due to Support Settlement

Member ab



$$FEM^{s}_{ab} = FEM^{s}_{ba} = \frac{-6EK_{ab}\Delta_{ab}}{l}$$

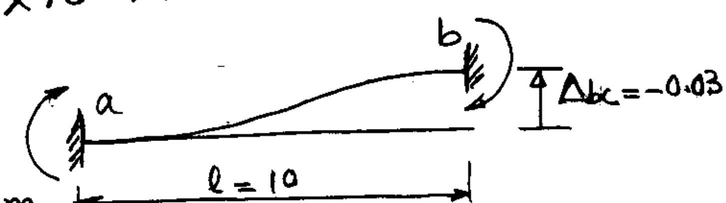
$$K_{ab} = \left(\frac{I}{L}\right)_{ab} = \frac{2000 \times 10^{-6}\, m^4}{10\, m} = 2000 \times 10^{-7}\, m^3$$

$$FEM^{s}_{ab} = FEM^{s}_{ba} = \frac{-6 \times 200 \times 10^{9} \times 2000 \times 10^{-7} \times (0.03)}{10}$$

$$= -720 \times 10^{3}\ N\text{-}m = -720\ KN\text{-}m$$

Member bc

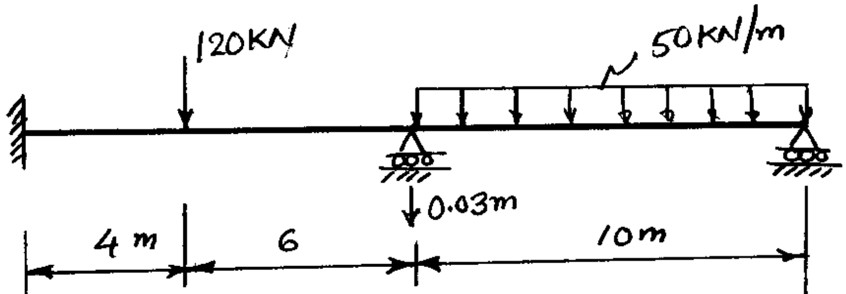
$$K_{bc} = \left(\frac{I}{L}\right)_{bc} = 2000 \times 10^{-7}\, m^3$$



$$FEM^{s}_{bc} = FEM^{s}_{bc} = +720\ KN\text{-}m$$

Moment Distribution Method

Example- Support Settlement



Distribution Factor	—	0.5	0.5	1.0
Carry Over Factor	—	0.5	0.5	0.5
FEM (load)	−172.8	+115.2	−416.7	+416.7
FEM (Settle)	−720.0	−720.0	+720.0	+720.0
	−892.8	−604.8	+303.3	+1136.7
			−568.4 ←	−1136.7
	+217.5 ←	+434.9	+434.9 →	+217.5
			−108.7 ←	−217.5
	+27.2 ←	+54.4	+54.4 →	+27.2
			−13.6 ←	−27.2
	+3.4 ←	+6.8	+6.8 →	+3.4
			−1.7 ←	−3.4
	+0.5 ←	+0.9	+0.9	
Final Moments KN-m	−644.2	−107.8	+107.9	0

Moment Distribution Method

"Cross-Carry-Over" Variant

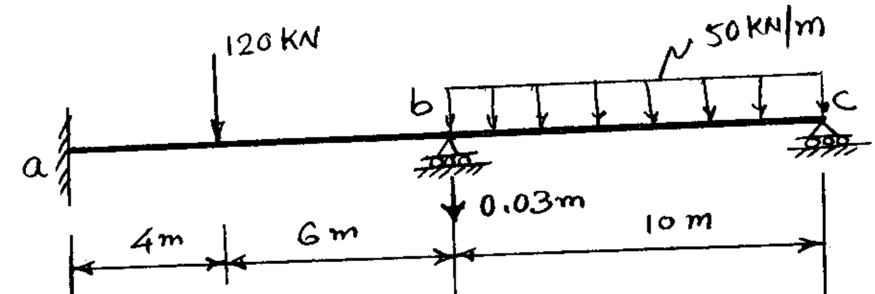
A variant of the Moment Distribution Method is the "Cross-Carry-Over" Moment-Distribution Method, in which several joints are released simultaneously, then balanced, and then multiple carry-over operations are carried out. The procedure does not have definite physical interpretation. However, it exhibits mathematical convergence to exact solution. The method takes its name from the cross-pattern of the carry-over arrows.

Example Problem

Given:

E = 200 GPa

Support b settles by 0.03 m



	a	b (ba)	b (bc)	c
Distribution Factor	—	0.5	0.5	1.0
Carry Over Factor	—	0.5	0.5	0.5
FEM (LOAD)	−172.8	+115.2	−416.7	+416.7
FEM (Settle)	−720.0	−720.0	+720.0	+720.0
Dist Cycle	—	+150.8	+150.8	−1136.7
Carry Over Dist.	+75.4		−568.4	+75.4
	—	+284.2	+284.2	−75.4
	+142.1		−37.7	+142.1
	—	+18.9	+18.9	−142.1
	+9.5		−71.1	+9.5
		+35.5	+35.5	−9.5
	+17.8		−4.8	+17.8
	—	+2.4	+2.4	−17.8
	+1.2		−8.9	+1.2
	—	+4.5	+4.5	−1.2
	+2.3		−0.6	+2.3
		+0.3	+0.3	−2.3
Final Moments	−644.5	−108.5	+108.4	0