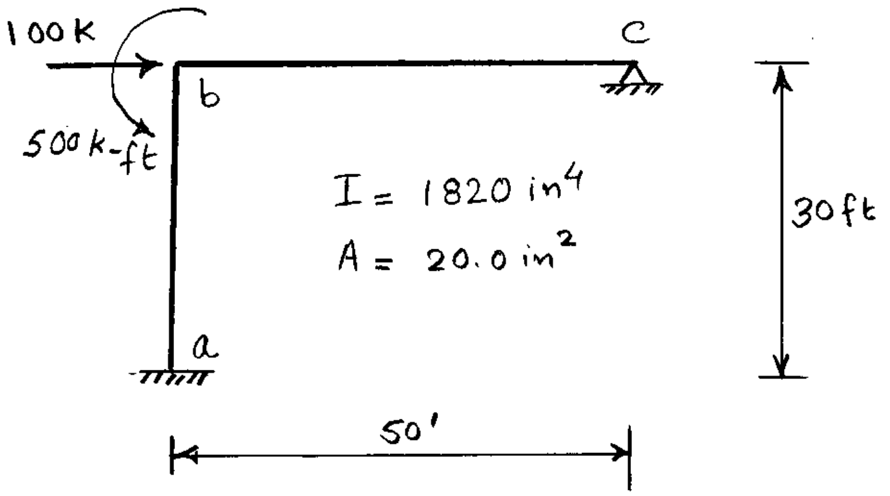
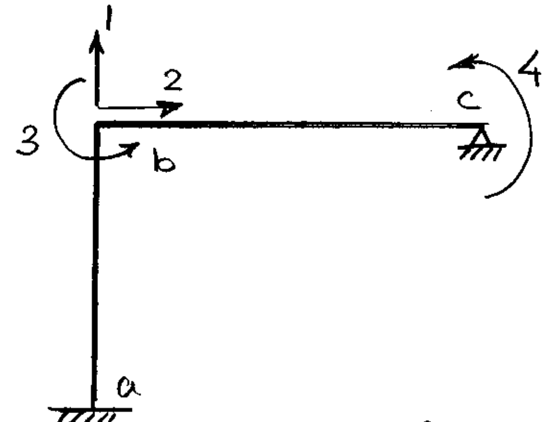
Matrix Analysis/ Stiffness Method

Example Problem



Determine the member end Forces for the frame Shown above



Structure Degrees of Freedom

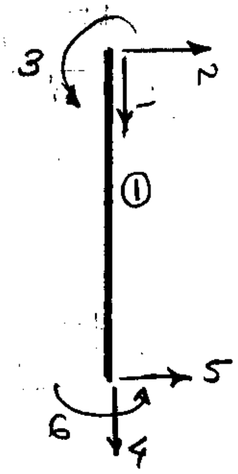
Member Stiffness matrices

Member ab

$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ F_4 \\ F_5 \\ M_6 \end{Bmatrix} = EI \begin{bmatrix} \frac{A}{LI} & 0 & 0 & -\frac{A}{L} & 0 & 0 \\ 0 & \frac{12}{L^3} & \frac{6}{L^2} & 0 & -\frac{12}{L^3} & \frac{6}{L^2} \\ 0 & \frac{6}{L^2} & \frac{4}{L} & 0 & -\frac{6}{L^2} & \frac{2}{L} \\ \frac{-A}{LI} & 0 & 0 & \frac{A}{LI} & 0 & 0 \\ 0 & -\frac{12}{L^3} & -\frac{6}{L^2} & 0 & \frac{12I}{L^3} & -\frac{6}{L^2} \\ 0 & \frac{6}{L^2} & \frac{2}{L} & 0 & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \theta_3 \\ \Delta_4 \\ \Delta_5 \\ \theta_6 \end{Bmatrix}$$

Matrix Analysis - Stiffness Method

Member ab or 1



$$\frac{A}{LI} = \frac{20/144 \times 12^4}{30 \times 1820} = 0.052747 \ ft^{-3}$$

$$\frac{12}{L^3} = \frac{12}{(30)^3} = 0.000,444 \ ft^{-3}$$

$$\frac{6}{L^2} = \frac{6}{(30)^2} = 0.00,667 \ ft^{-3}$$

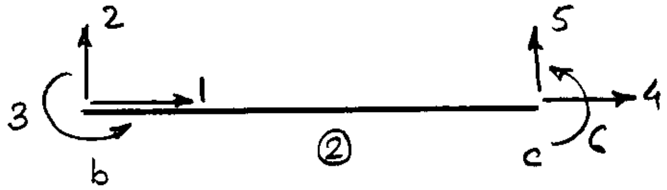
$$\frac{4}{L} = \frac{4}{30} = 0.133,333 \ ft^{-3}$$

$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \end{Bmatrix} = EI \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.052747 & 0 & 0 \\ 0 & 0.000,444 & 0.00,667 \\ 0 & 0.00,667 & 0.133,333 \end{bmatrix} \end{array} \begin{Bmatrix} \Delta_4 \\ \Delta_5 \\ \theta_6 \end{Bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array}$$

Destination Array

$$DA① = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$$

Member bc or 2



$$\frac{A}{LI} = \frac{20/144 \times 12^4}{50 \times 1820} = 0.031,7$$

$$\frac{12}{L^3} = \frac{12}{(50)^3} = 0.000,096$$

$$\frac{6}{L^2} = \frac{6}{(50)^2} = 0.002,40$$

$$\frac{4}{L} = \frac{4}{50} = 0.080,0$$

Matrix Analysis / Stiffness Method

Member bc – ②

$$
\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \\ M_6 \end{Bmatrix} = EI
\begin{array}{c}
\begin{array}{cccc} 2 & 1 & 3 & 6 \end{array} \\
\begin{bmatrix}
0.0317 & 0 & 0 & 0 \\
0 & 0.000{,}096 & -0.00240 & 0.00{,}240 \\
0 & -0.00240 & 0.080{,}0 & 0.040{,}0 \\
0 & 0.00{,}240 & 0.040{,}0 & 0.080{,}0
\end{bmatrix}
\end{array}
\begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \theta_6 \end{Bmatrix}
\begin{array}{c} 2 \\ 1 \\ 3 \\ 6 \end{array}
$$

Destination Array

$$
DA_{②} = \begin{Bmatrix} 2 \\ 1 \\ 3 \\ 6 \end{Bmatrix}
$$

Structure Stiffness Matrix

$$
[K] = EI
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 3 & 6 \end{array} \\
\begin{bmatrix}
0.052747 + 0.000{,}096 & 0 & 0 - 0.00240 & 0.00240 \\
0 & 0.000{,}444 + 0.0317 & 0.00{,}667 & 0 \\
0 - 0.00{,}240 & 0.00{,}667 & 0.133{,}333 + 0.080{,}0 & 0.040{,}0 \\
0.00240 & 0 & 0.040{,}0 & 0.080{,}0
\end{bmatrix}
\end{array}
\begin{array}{c} 1 \\ 2 \\ 3 \\ 6 \end{array}
$$

$$
[K]_{\text{Structure}} = EI
\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 3 & 6 \end{array} \\
\begin{bmatrix}
0.052{,}843 & 0 & -0.0024 & 0.0024 \\
0 & 0.032144 & 0.00667 & 0 \\
-0.0024 & 0.00667 & 0.213333 & 0.040 \\
0.0024 & 0 & 0.040 & 0.080
\end{bmatrix}
\end{array}
\begin{array}{c} 1 \\ 2 \\ 3 \\ 6 \end{array}
$$

Matrix Analysis – Stiffness Method

Structure Equilibrium Equation

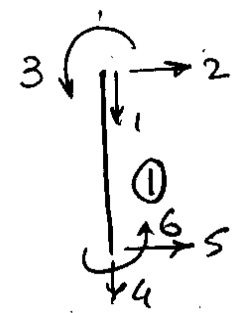
$$[K]\{\Delta\} = \{P\}$$

$$EI \begin{bmatrix} 0.052843 & 0 & -0.0024 & 0.0024 \\ 0 & 0.032144 & 0.00667 & 0 \\ -0.0024 & 0.00667 & 0.213333 & 0.040 \\ 0.0024 & 0 & 0.040 & 0.080 \end{bmatrix} \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \theta_3 \\ \theta_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 100 \\ 500 \\ 0 \end{Bmatrix}$$

$$\{\Delta\} = [K]^{-1}\{P\}$$

$$\Rightarrow \{\Delta\} = \begin{Bmatrix} \Delta_1 \\ \Delta_2 \\ \theta_3 \\ \theta_6 \end{Bmatrix} = \frac{1}{EI} \begin{Bmatrix} 170.544 \\ 2592.576 \\ 2499.94 \\ -1255.09 \end{Bmatrix}$$

Member End Forces



$$\begin{Bmatrix} F_4 \\ F_5 \\ M_6 \end{Bmatrix} = \frac{EI}{EI} \begin{bmatrix} -0.052747 & 0 & 0 \\ 0 & -0.000444 & -0.00667 \\ 0 & 0.00667 & 0.06667 \end{bmatrix} \begin{array}{l} \begin{Bmatrix} 170.544 \\ 2592.58 \\ 2499.94 \end{Bmatrix} \end{array} \begin{array}{l} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{array}$$

$$= \begin{Bmatrix} -9.0 \\ -17.83 \\ 183.96 \end{Bmatrix} \begin{array}{l} \text{Kips} \\ \\ \text{k-ft} \end{array}$$

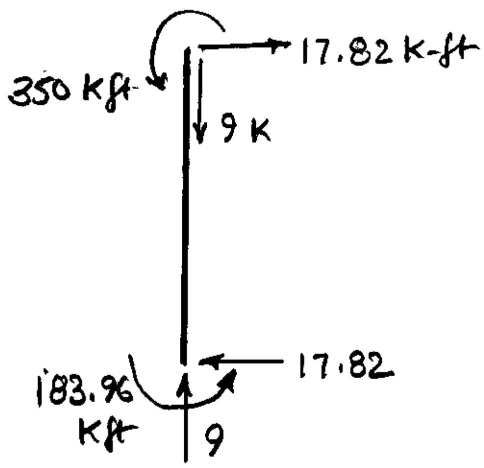
Matrix Analysis – Stiffness Method

Member End Forces

member ab – 1

$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \end{Bmatrix} = \frac{EI}{EI} \begin{bmatrix} 0.052747 & 0 & 0 \\ 0 & 0.000444 & 0.00667 \\ 0 & 0.00667 & 0.133,333 \end{bmatrix} \begin{array}{c} \begin{Bmatrix} 170.544 \\ 2592.58 \\ 2499.94 \end{Bmatrix} \end{array} \begin{array}{l} \Delta_1 \\ \Delta_2 \\ \theta_3 \end{array}$$

$$= \begin{Bmatrix} 9.0 \\ 17.82 \\ 350.62 \end{Bmatrix}$$



Member bc – 2

$$\begin{Bmatrix} F_1 \\ F_2 \\ M_3 \end{Bmatrix} = \frac{EI}{EI} \begin{bmatrix} 0.0317 & 0 & 0 & 0 \\ 0 & 0.000096 & 0.0024 & 0.0024 \\ 0 & 0.0024 & 0.080 & 0.040 \end{bmatrix} \begin{Bmatrix} 2592.58 \\ 170.544 \\ 2499.94 \\ -1255.09 \end{Bmatrix} \begin{array}{l} \Delta_1 \\ \Delta_2 \\ \theta_3 \\ \theta_6 \end{array}$$

$$= \begin{Bmatrix} 82.18 \\ 3.0 \\ 150.2 \end{Bmatrix} \begin{array}{l} k \\ k \\ k\text{-ft} \end{array}$$

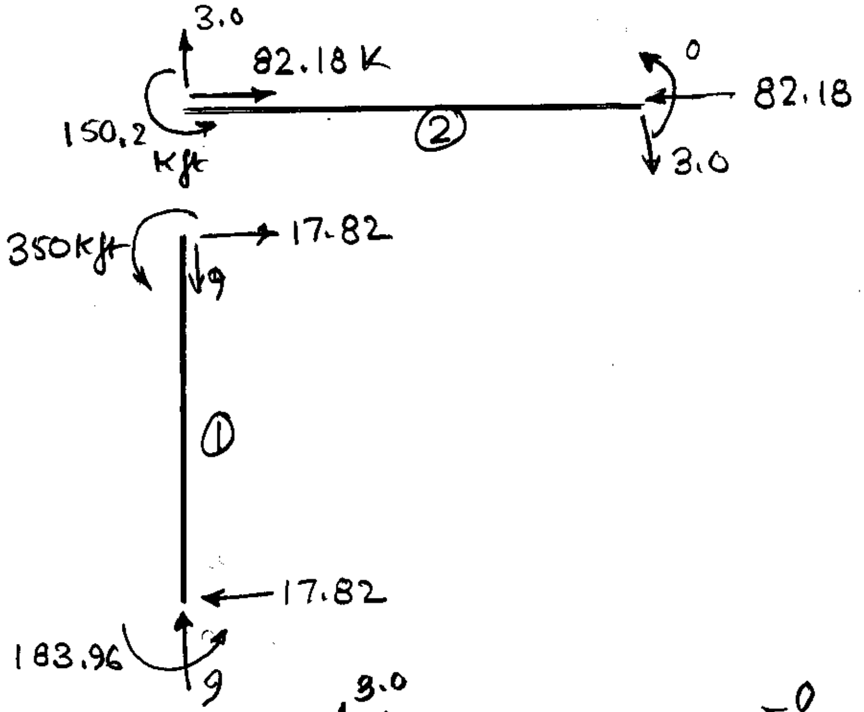
Matrix Analysis - Stiffness Method

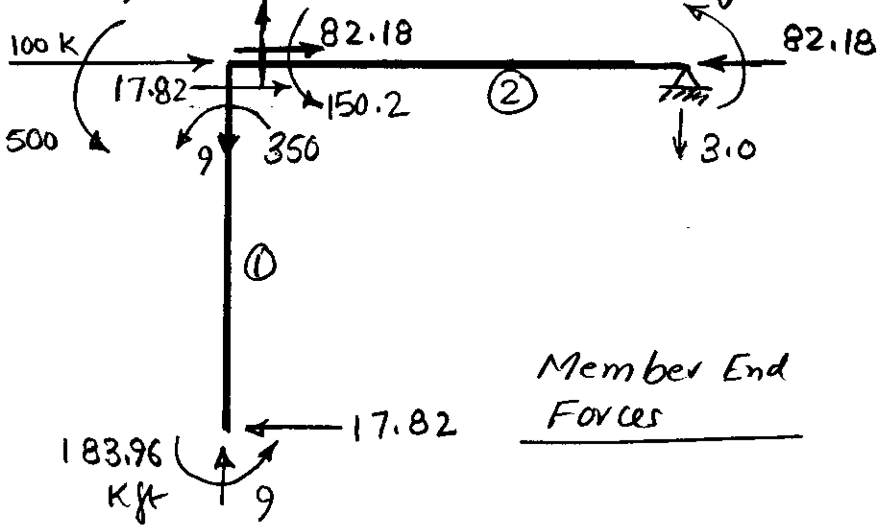
Member End Forces

Member bc - ②

$$\begin{Bmatrix} F_4 \\ F_5 \\ M_6 \end{Bmatrix} = \frac{EI}{EI} \begin{bmatrix} -0.0317 & 0 & 0 & 0 \\ 0 & -0.000096 & -0.0024 & -0.0024 \\ 0 & 0.0024 & 0.0400 & 0.080 \end{bmatrix} \begin{array}{c} \begin{Bmatrix} 2592.58 \\ 170.544 \\ 2499.94 \\ -1255.09 \end{Bmatrix} \end{array} \begin{array}{c} \Delta_1 \\ \Delta_2 \\ \theta_3 \\ \theta_6 \end{array}$$

$$\begin{Bmatrix} F_4 \\ F_5 \\ M_6 \end{Bmatrix} = \begin{Bmatrix} -82.18 \\ -3.0 \\ -0.0003 \end{Bmatrix} = \begin{Bmatrix} -82.18 \\ -3.0 \\ \sim 0 \end{Bmatrix}$$



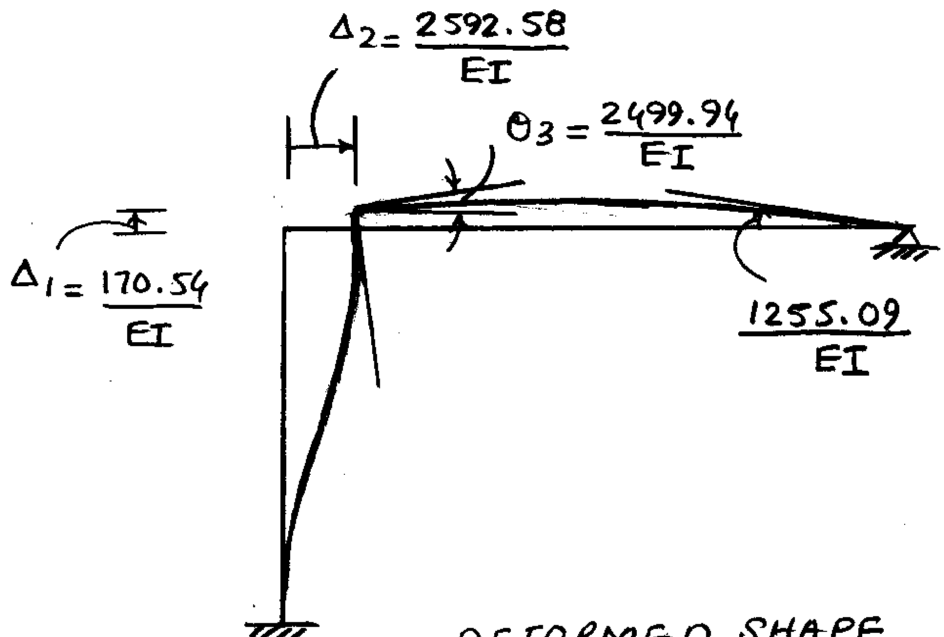


Note: error in Vertical Reation at member ① due to truncation/ roundoff.

Member End Forces

Matrix Analysis – Stiffness Method

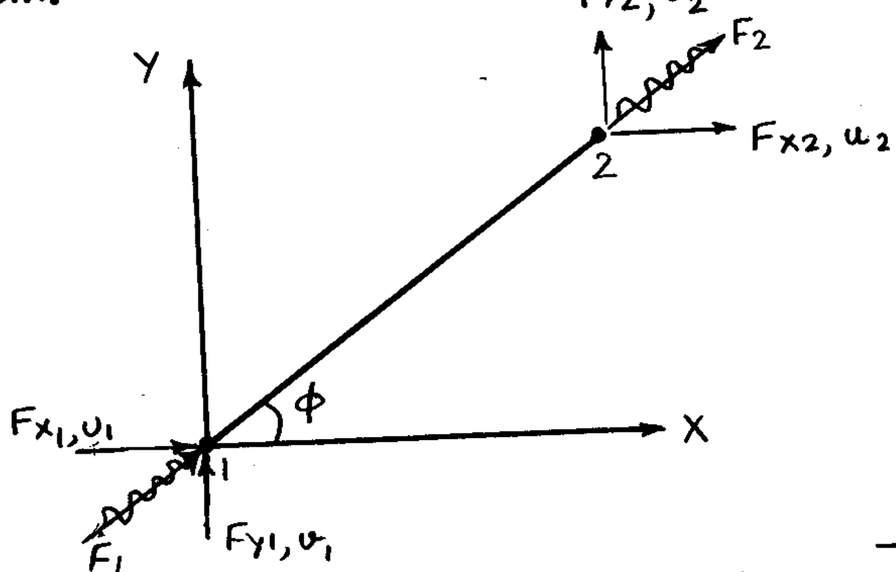
Deformed Shape

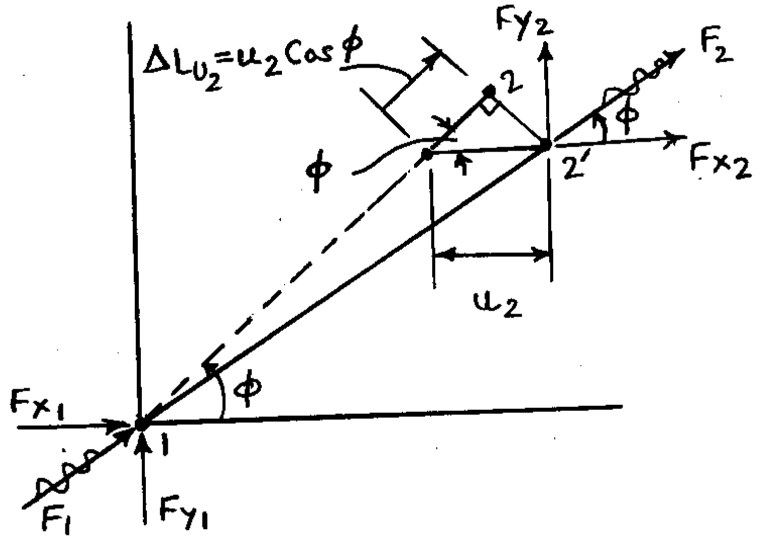


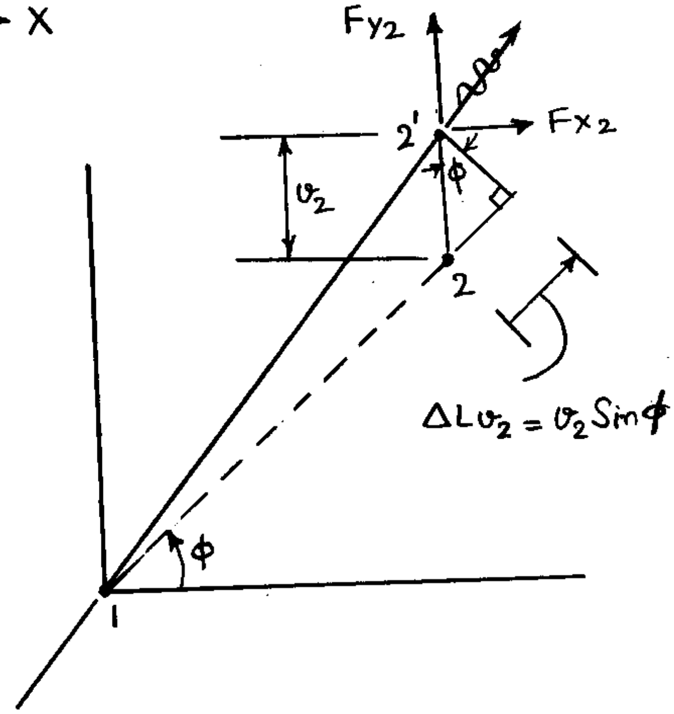
DEFORMED SHAPE

Stiffness Matrix of Truss Element in Global Coordinates

Consider the truss member shown below that makes an angle ϕ with the X, Y Global Coordinates. The element has two degrees of freedom in the element local coordinates. However, it has Four (4) Global Coordinate degrees of freedoms in the System.







Assuming that small displacement Theory is valid we generate the member stiffness matrix from 1st Principles. If Node 2 is given a small displacement in x dir u_2 Then change in bar length is:

$$\Delta L_{u_2} = u_2 \cos\phi$$

Stiffness Matrix of a Truss Element in Global Coordinates

The Axial resultant Force on Bar $= F_2 = \frac{EA}{L} \Delta L_{u_2} = \frac{EA \cos\phi}{L} . u_2$

The Four Forces in Global Coordinates can now be determined by equilibrium.

$$\left.\begin{aligned} F_{x2} &= -F_{x1} = F_2 \cos\phi = \frac{EA}{L} \cos^2\phi \; . u_2 \\ F_{y2} &= -F_{y1} = F_2 \sin\phi = \frac{EA}{L} \sin\phi \cos\phi \, . u_2 \end{aligned}\right\} \quad (2)$$

Similarly, for a small displacement at Node 2 in the y-direction equal to v_2, The change in length of the bar is:

$$\Delta L_{v_2} = v_2 \sin\phi$$

Axial Resultant Force in Bar $= F_2 = \frac{EA}{L} \Delta L_{v_2} = \frac{EA \sin\phi}{L} v_2$

The Forces in Global Coordinates are:

$$\left.\begin{aligned} F_{x2} &= -F_{x1} = F_2 \cos\phi = \frac{EA}{L} \sin\phi \cos\phi \, . v_2 \\ F_{y2} &= -F_{y1} = F_2 \sin\phi = \frac{EA}{L} \sin^2\phi \, . v_2 \end{aligned}\right\} \quad (4)$$

In the same manner displacements can be imposed on node 1 and corresponding Forces in Global Coordinates determined.

Stiffness Matrix of a Truss Element in Global Coordinates

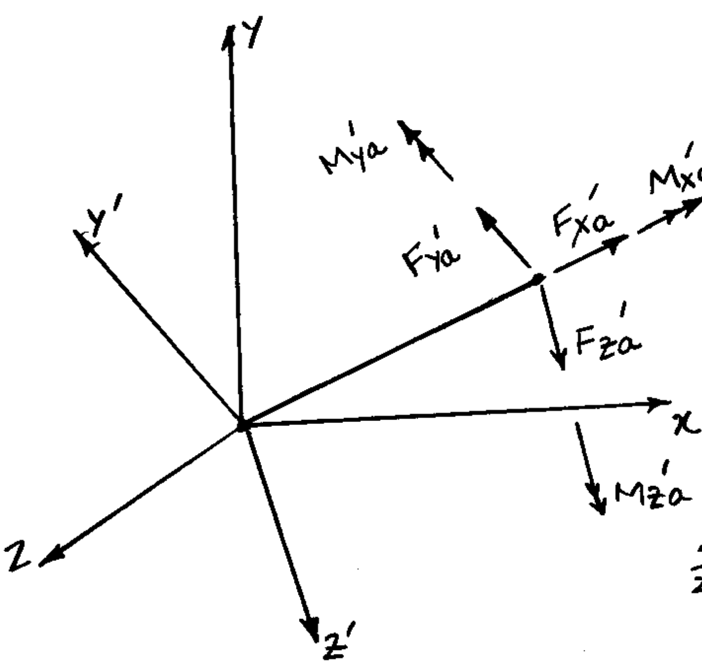
The Final Truss Element Stiffness Matrix in Global Coordinates is then as Follows:

$$\begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{x2} \\ F_{y2} \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} \cos^2\phi & \sin\phi\cos\phi & -\cos^2\phi & -\sin\phi\cos\phi \\ \sin\phi\cos\phi & \sin^2\phi & -\sin\phi\cos\phi & -\sin^2\phi \\ -\cos^2\phi & -\sin\phi\cos\phi & \cos^2\phi & \sin\phi\cos\phi \\ -\sin\phi\cos\phi & -\sin^2\phi & \sin\phi\cos\phi & \sin^2\phi \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix} \quad \text{— (5)}$$

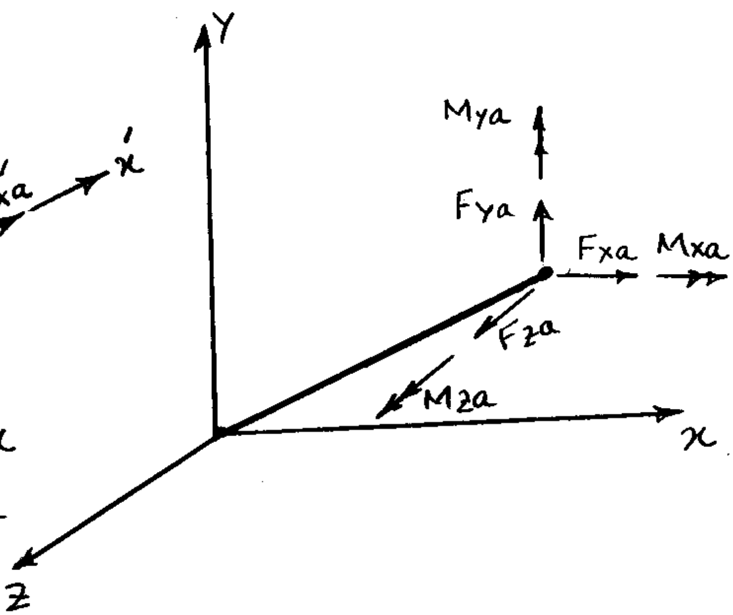
Note: that in the above stiffness matrix cols (3) and (4) can be written by referring to Eqs (2) & (4) on the previous page. Cols (1) and (2) can then be written down directly as Col (1) is -ive of Col. (3) and Col (2) is -ive of Col (4)

Coordinate Transformation & Stiffness Matrix Transformation.

In a structural System there are usually many Structural members, each having a different orientation and different directions of degrees of freedom in the member "Local Coordinate System". AT the structure joints/nodes where the various members meet, the joint/nodal degrees of freedom are defined in the "Global Coordinate System"

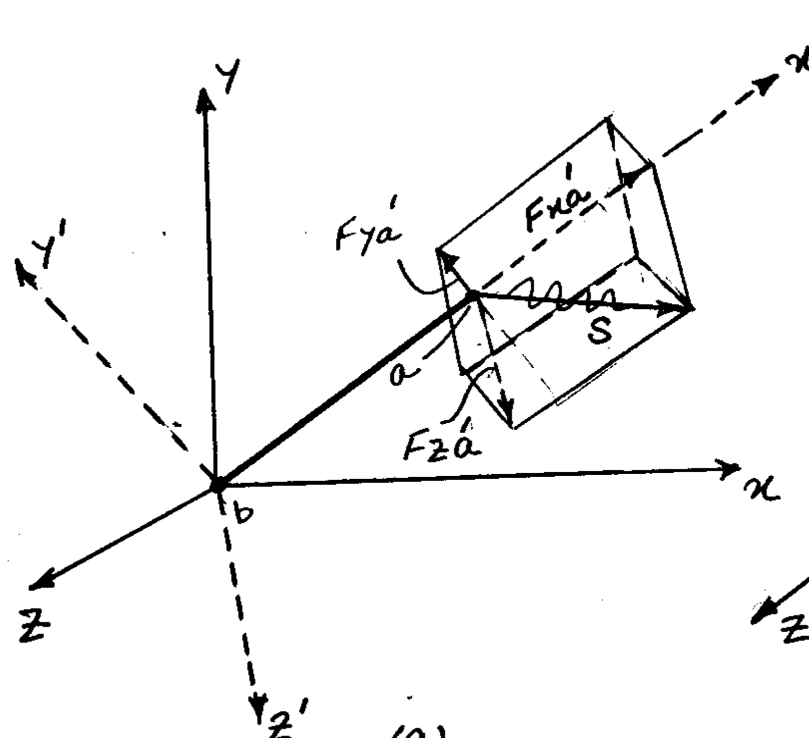


Structural Member in Local Coordinate System with Forces and Degrees of Freedom in Local Coords



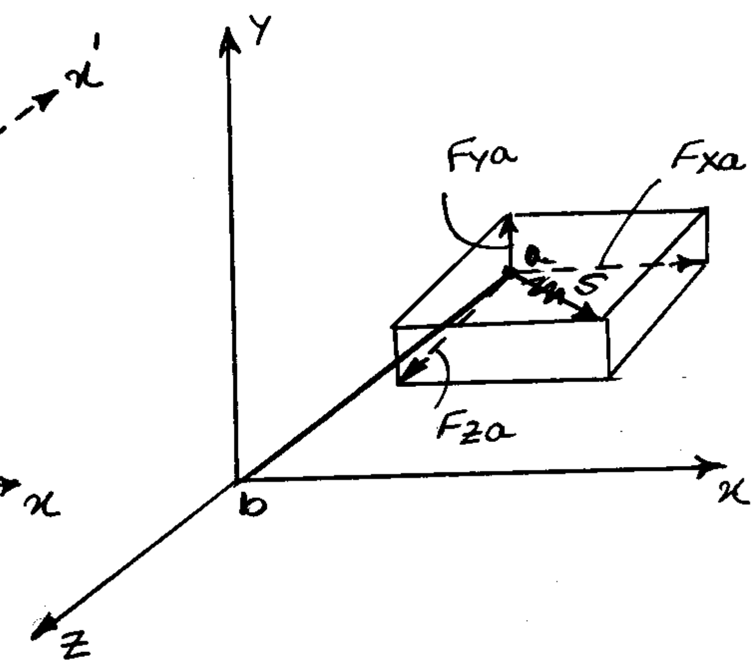
Structural Member In Global Coordinate System with Freedoms in Global Coordinate System

Coordinate Transformation & Stiffness Matrix Transformation.



(a)

Force "S" Expressed in Local Coordinates



(b)

Force "S" Expressed in Global Coordinates

Consider a Force "S" that is expressed in Local coordinates in Fig.(a) and expressed in Global Coordinates in Fig (b). The force components in Local Coords are: $F_{xa'}, F_{ya'}, F_{za'}$ and in Global Coords are: F_{xa}, F_{ya}, F_{za} respectively

Then the forces in Local Coords are related to the force components in Global Coords by the following relation:

$$\left.\begin{aligned} F_{xa'} &= F_{xa}\cos\alpha_{x'} + F_{ya}\cos\beta_{x'} + F_{za}\cos\delta_{x'} \\ F_{ya'} &= F_{xa}\cos\alpha_{y'} + F_{ya}\cos\beta_{y'} + F_{za}\cos\delta_{y'} \\ F_{za'} &= F_{xa}\cos\alpha_{z'} + F_{ya}\cos\beta_{y'} + F_{za}\cos\delta_{z'} \end{aligned}\right\} \text{—①}$$

$\cos\alpha_{x'}, \cos\beta_{x'}$ --- $\cos\delta_{z'}$ are cosines of angles between the Local Coordinates and the Global Coordinates simply referred to as "Direction Cosines"

Coordinate Transformation & Stiffness Matrix Transformation

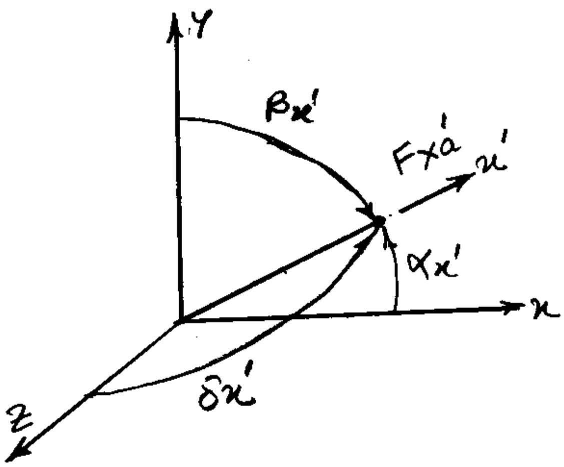
The Direction Cosines $\cos\alpha x'$, $\cos\beta x'$ ... $\cos\delta z'$ in the Transformation Equation ① are defined as:

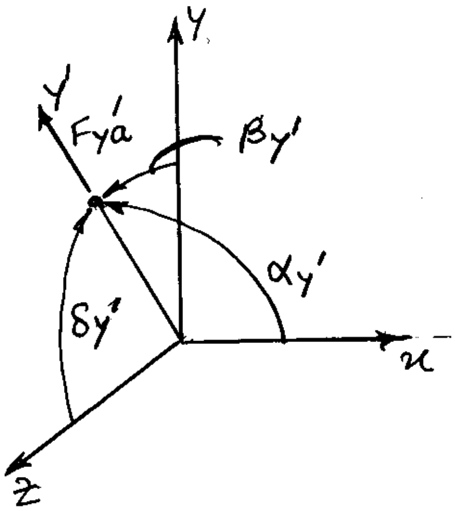
$\cos\alpha x' = l_1 =$ Cos of angle between x' and x

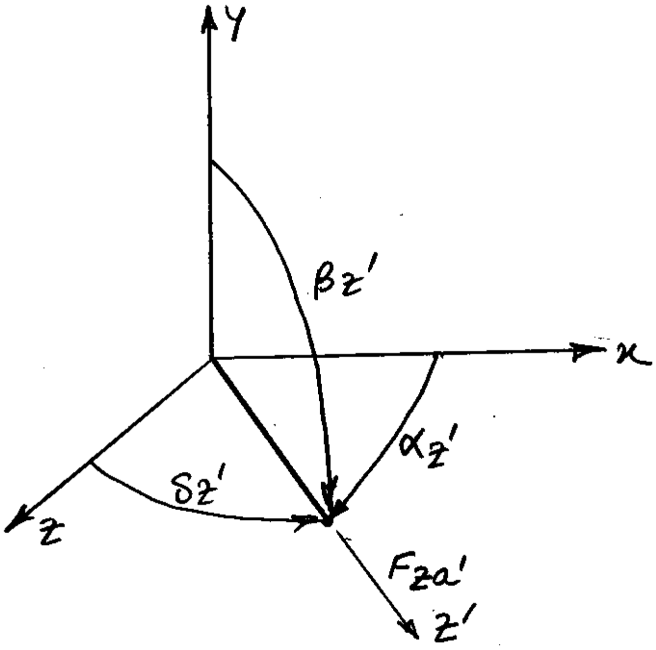
$\cos\beta x' = m_1 =$ " " " " x' and y

$\cos\delta x' = n_1 =$ " " " " x' and z

The Direction Cosines are graphically described in the Figures below:







Direction Cosines are also sometimes expressed in the following Tabular Format

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

Where $l_1 = \cos x'x$, $m_1 = \cos x'y$
$n_1 = \cos x'z$ etc

Coordinate Transformation & Stiffness Matrix Transformation.

Equation ① relating Forces in Local Coordinates to Forces in Global coordinates can be written in matrix form as:

$$\begin{Bmatrix} F_{xa}' \\ F_{ya}' \\ F_{za}' \end{Bmatrix} = \begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix} \begin{Bmatrix} F_{xa} \\ F_{ya} \\ F_{za} \end{Bmatrix} \quad \text{—— ②}$$

or in short Matrix Form

$$\{F_F'\} = [\lambda] \{F_F\} \quad \text{—— ③}$$

The Matrix $[\lambda]$ of direction cosines is called a "Rotation/Transformation Matrix"

Recall that som of squares of direction cosines for any axis is unity, therefore:

$$\left.\begin{aligned} l_1^2 + m_1^2 + n_1^2 &= 1 \\ l_2^2 + m_2^2 + n_2^2 &= 1 \\ l^3 + m_3^2 + n_3^2 &= 1 \end{aligned}\right\} \quad \text{—— (4-a)}$$

or in compact form:

$$\lfloor l_i \;\; m_i \;\; n_i \rfloor \cdot \begin{Bmatrix} l_i \\ m_i \\ n_i \end{Bmatrix} = 1 \quad \text{—— (4-b)}$$

Also recall that the Local Coordinate Axes x', y', z' are orthogonal Axes, dot product of vectors along x' and y', x' and z' and y' and z' is zero, i.e.

$$[l_i \quad m_i \quad n_i] \cdot \begin{Bmatrix} l_j \\ m_j \\ n_j \end{Bmatrix} = 0 \quad , \text{for } i \neq j \qquad \text{— (5-a)}$$

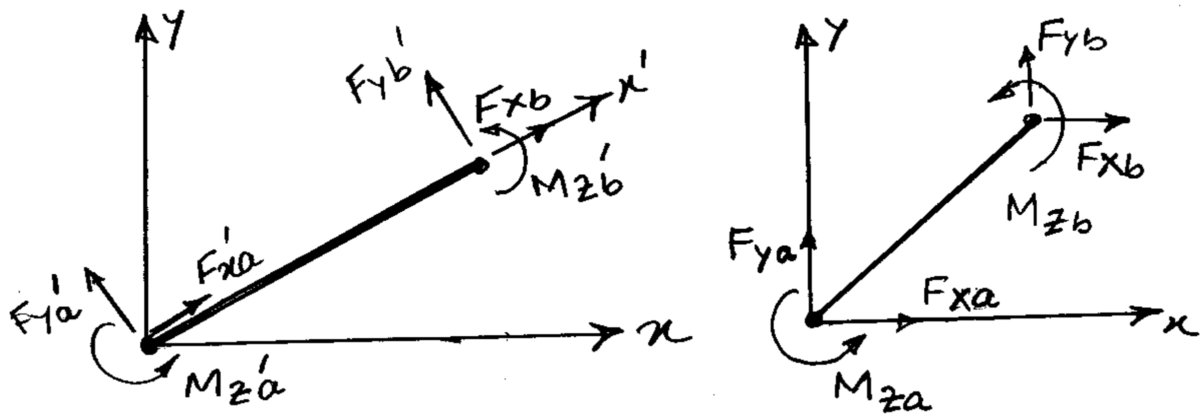
or in expanded form:

$$\left.\begin{aligned} l_1 l_2 + m_1 m_2 + n_1 n_2 &= 0 \\ l_1 l_3 + m_1 m_3 + n_1 n_3 &= 0 \\ l_2 l_3 + m_2 m_3 + n_2 n_3 &= 0 \end{aligned}\right\} \qquad \text{— (5-b)}$$

Equations (4) & (5) indicate that the matrix of Direction Cosines is an "orthogonal matrix" i.e

$$\boxed{[\gamma]^{-1} = [\gamma]^T} \qquad \text{— (6)}$$

Since forces and moments in the member Local Coordinates (primed Coords) are vector quantities that can be expressed in terms of Forces and Moments in Global Coordinates through the Transformation Matrix, we can write the following relation between forces in Local Coords and forces in Global Coords.



Coordinate Transformation & Stiffness Matrix Transformation.

$$\begin{Bmatrix} F_{xa}' \\ F_{ya}' \\ M_{za}' \\ \hline F_{xb}' \\ F_{yb}' \\ M_{zb}' \end{Bmatrix} = \left[\begin{array}{c|c} [\gamma]_{3\times3} & 0 \\ \hline 0 & [\gamma]_{3\times3} \end{array}\right] \begin{Bmatrix} F_{xa} \\ F_{ya} \\ M_{za} \\ \hline F_{xb} \\ F_{yb} \\ M_{zb} \end{Bmatrix} \quad \text{—— (7-a)}$$

or in compact Form

$$\{F'\} = [T]\{F\} \quad \text{—— (7-b)}$$

where,

$$[T] = \text{Transformation Matrix}$$

$$[T]^{-1} = [T]^{T} \quad \text{—— (8)}$$

Transformation of Stiffness Matrix

If the Force-Displacement Relations in Local Coords is as follows:

$$\{F'\} = [K']\{\Delta'\} \quad \text{—— (9)}$$

Then using (7-b) we have

$$[T]\{F\} = [K'][T]\{\Delta\} \quad \text{—— (10)}$$

$$\{F\} = \underbrace{[T]^{T}[K'][T]}_{[K] \text{ Global Stiffness}}\{\Delta\} \quad \text{—— (11)}$$

$$\Rightarrow \{F\} = [K]\{\Delta\}$$

Where,

$$\boxed{[K] = \text{Global Stiffness Matrix} = [T]^{T}[K'][T]} \quad \text{—— (12)}$$